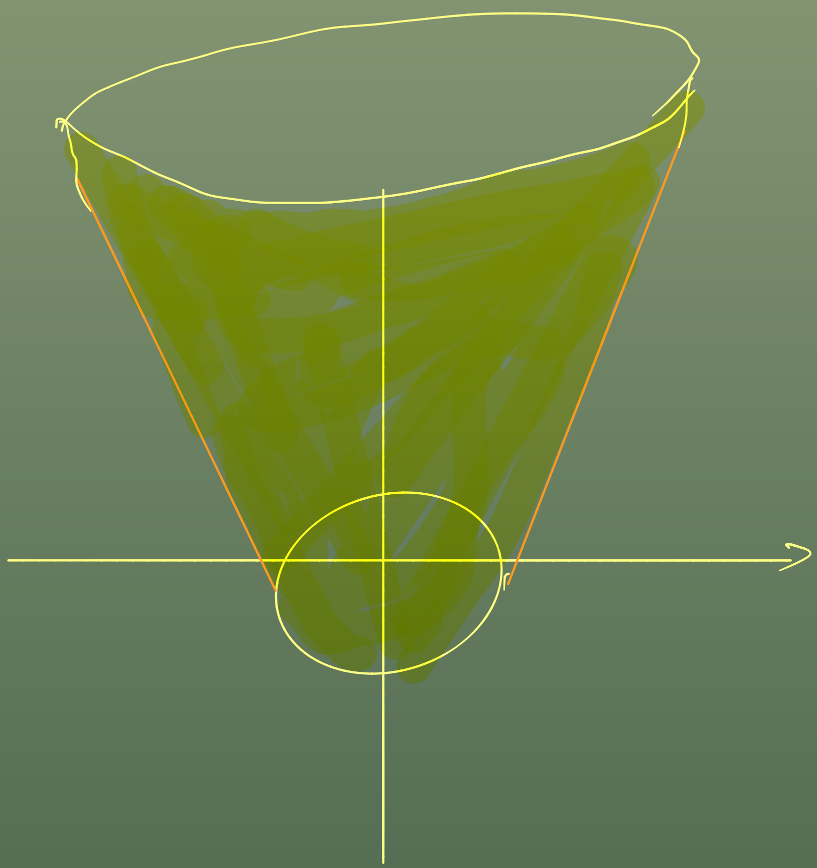
Green Functions.



Week 1

Main Stream.

一、 Burger's Equation.

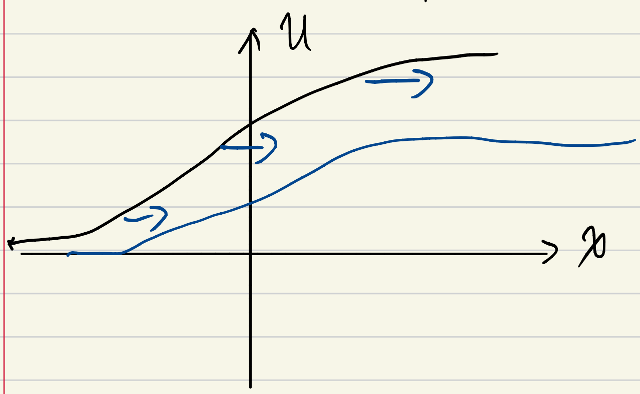
$$u_t + u u_x = u_{xx}, \quad x \in \mathbb{R},\ t > 0$$

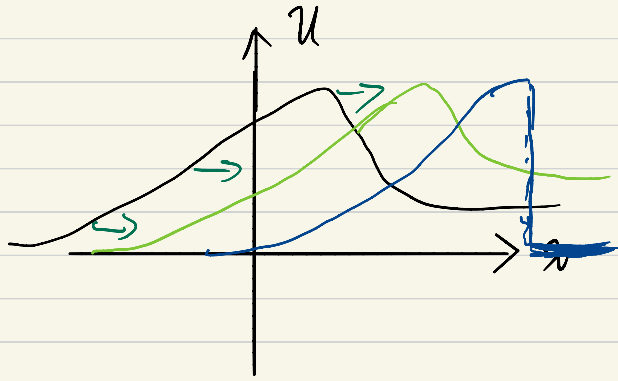
- First,

$$u_t + u u_x = 0, \quad \text{hyperbolic equation}$$

Transport eq.: $u_t + \underline{c}\, u_x = 0$

e.g.





Then, $u_t + \left(\frac{u^2}{2}\right)_x = u_{xx}$

Introduce: $B_x = u$, $\Rightarrow B_{xt} + \left(\frac{B_x^2}{2}\right)_x = (B_{xx})_x \quad (*)$

Integrate $(*)$ in x, assume $B_t = 0$ in the infinity,

$$B_t + \frac{B_x^2}{2} = B_{xx}$$

Then, take $B = -2\log\phi$, $B_t = -2\frac{\phi_t}{\phi}$, $B_x = -2\frac{\phi_x}{\phi}$

$$B_x^2 = 4 \cdot \left(\frac{\phi_x}{\phi}\right)^2$$

$$\& \quad B_{xx} = -2\,\frac{\phi_{xx}\phi - \phi_x^2}{\phi^2}$$

Note:

$$-2\frac{\phi_t}{\phi} + 2\frac{\phi_x^2}{\phi^2} = -2\frac{\phi_{xx}}{\phi} + \frac{2\phi_x^2}{\phi^2}$$

$\Rightarrow \phi_t = \phi_{xx}$ heat equation!

Thus, $\phi(x,t) = \int_{-\infty}^{+\infty} \frac{e^{\frac{-(x-y)^2}{4t}}}{\sqrt{4\pi t}} \phi(y,0)\, dy = \underline{\underline{K}} * \phi(x,0)$

Green function of heat equation.

Since $\underline{\phi(x,t) = e^{-\frac{1}{2}B(x,t)}}$

& $B_x = u$,

then

Hopf-Cole transform.

$$u(x,t) = \int_{-\infty}^{x} u(y,t)\, dy,$$

initial data.

$$\phi(x,t) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\int_{-\infty}^{x} u(z,0)\, dz} \cdot \frac{e^{\frac{-(x-y)^2}{4t}}}{\sqrt{4\pi t}}\, dy$$

$$u(x,t) = \partial_x B = \partial_x (-2) \log \phi = -\frac{2\phi_x}{\phi}$$

This can be solved by

Next, want to use a new method to do it!

Preliminaries:

- Fix Point Theorem:

Picard's iteration, $\begin{cases} y' = f(t,y) \\ y(0) = y_0 \end{cases}$

$$y_1(t) = y_0 + \int_0^t f(z, y_0)\, dz$$

$$y_2(t) = y_0 + \int_0^t f(z, y_1(z))\, dz$$

$$\vdots$$

$$y_n(t) = y_0 + \int_0^t f(z, y_{n-1}(z))\, dz$$

Want to have ratio test:

$$y_n = y_0 + \int_0^t f(z, y_{n-1})\, dz$$

$$- \quad y_{n-1} = y_0 + \int_0^t f(z, y_{n-2})\, dz$$

$$y_n - y_{n-1} = \int_0^t f(z, y_{n-1}) - f(z, y_{n-2})\, dz$$

MVT: $|f(z, y_{n-1}) - f(z, y_{n-2})| \leq |y_{n-1} - y_{n-2}| \cdot \max_{\xi \in (y_{n-1}, y_{n-2})} |f_y(z, \xi)| \leq M$ assume

Then,

$$|y_n(t) - y_{n-1}(t)| \leq M \int_0^t |y_{n-1}(z) - y_{n-2}(z)|\, dz$$

Metric space:

Fix t_0, $\quad \|y_n - y_{n-1}\| = \sup_{z \in [0, t_0]} |y_n(z) - y_{n-1}(z)|$

Let $t < t_0$

$$\Rightarrow \underbrace{|y_n(t) - y_{n-1}(t)|}_{\sup_{t \in [0,t_0)}} \leq \underbrace{M}_{\sup_{t \in [0,t_0)}} \int_0^t \|y_{n-1} - y_{n-2}\|\, dz \leq M t_0 \|y_{n-1} - y_{n-2}\|$$

by

$$\Rightarrow \|y_n - y_{n-1}\| \leq M t_0 \|y_{n-1} - y_{n-2}\|$$

How ? : By taking t_0 small

If choose $M t_0 < 1$, then $\lim_{n \to \infty} \|y_n - y_{n-1}\| = 0$

Moreover, $\|y_n - y_{n-1}\| \leq C_0 (M t_0)^n$ ——— (*)

Now, just need to choose t_0 at beginning.

Then, consider:

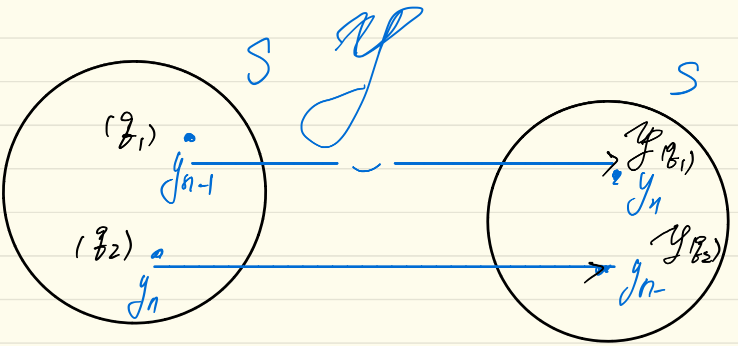
$y_1 + (y_2 - y_1) + \dots (y_n - y_{n-1})$ is convergent by (*)

Implying:

$\lim_{n \to \infty} y_n$ exists

Consider fixed point thm:

$$y_n(t) = y_0 + \int_0^t f(z, y_{n-1}(z))\,dz$$



$\|q_1 - q_2\|$ $\qquad$ $\|\mathcal{Y}(q_1) - \mathcal{Y}(q_2)\|$

If $\|\mathcal{Y}(q_1) - \mathcal{Y}(q_2)\| \le \alpha \|q_1 - q_2\|$, with $\alpha < 1$,

then there exists a fixed point q^*.

Fix point Thm: $\left\{ \begin{array}{l} f: S \to S, \text{ complete space } S. \\ f \text{ is } \alpha \text{ contract map.} \end{array} \right.$ $\Rightarrow \exists!$ fixed point.

Want to show it satisfies the condition of fix point thm:

- $y_0 \in S$, & $y_n \in \mathcal{Y}(y_{n-1})$ $\forall n \geq 1$

  $\sum_{n=1}^{\infty} \|y_n - y_{n-1}\|$ converges.

Proof:

$$\|y_n - y_{n-1}\| = \|\mathcal{F}(y_{n-1}) - \mathcal{F}(y_{n-2})\| < \alpha \|y_{n-1} - y_{n-2}\|$$

$$\Rightarrow \quad \|y_n - y_{n-1}\| < \alpha^{n-1} \|y_1 - y_0\|$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \|y_n - y_{n-1}\| < \frac{\|y_1 - y_0\|}{\alpha}$$

What is the Green function or fundamental solutions?

"ODE"

$$\begin{cases} \vec{y}' = A\vec{y} \\ \vec{y}(0) = \vec{y}_0 \end{cases} \quad \Rightarrow \quad e^{-At}(y' - Ay) = 0$$

$$\Rightarrow \quad (e^{-At} y)' = 0$$

$$\Rightarrow \quad e^{-At} y = c \quad , \quad c = y_0$$

$$\Rightarrow \quad y(t) = e^{At} y_0$$

Now, A is a matric

Q: What is e^{At}? $\qquad A = S^{-1} \Lambda S$

A: $\quad e^{A} = S^{-1} e^{\Lambda} S, \quad e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\cdots$

"PDE":

$$\partial_t u = \partial_x u$$

$$(a_0, a_1, \cdots) \in \mathbb{R}^{\infty}$$

By Talyer's expansion: $\quad u(x,t) = a_0(t) + a_1(t)x + \cdots a_n(t)x^n + \cdots$

$$\partial_x u = a_1(t) + 2a_2(t)x + \cdots n a_n(t) x^{n-1} + \cdots$$

$$(a_1, 2a_2, \cdots, n a_n, \cdots)$$

So,

$$\frac{\partial}{\partial t}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 2 & \cdots & \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & & & & \\ 0 & \cdots & & & \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

$$L^2(\mathbb{R}) = \{ f \mid \int_{-\infty}^{\infty} f^2_{(x)}\, dx < \infty \}.$$

$$(f, g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$ — Inner Product.

$$\|f\|^2 = (f, f)$$ — $\|\cdot\|$: Norm for $L^2(\mathbb{R})$

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

$$\|\cdot\|_{L^1} = \int_{-\infty}^{\infty} |f|\, dx$$

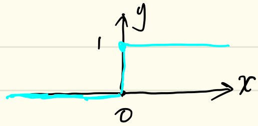
Fourier transform:

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-iyx} \cdot f(x)\, dx$$

Inverse Fourier transform:

$$f(x) = \frac{1}{2\pi}\int \hat{f}(y)\, e^{iyx}\, dy$$

$\delta(x) = \partial_x H(x)$: $\quad H(x) = \begin{cases} 1, & \text{if } x>0, \\ 0, & \text{if } x<0 \end{cases}$



$$\int_{-\infty}^{\infty} \delta(x)\,dx = \int_{-\infty}^{\infty} \partial_x H(x)\,dx = H\Big|_{-\infty}^{\infty} = 1-0 = 1$$

$f \in C_0(\mathbb{R})$: then $\displaystyle\int_{-\infty}^{\infty} \delta(x) f(x)\,dx$

$$= \int_{-\infty}^{\infty} \partial_x H(x)\cdot f(x)\,dx \overset{\text{I.B.P}}{=} -\int_{-\infty}^{\infty} H(x) f'(x)\,dx$$

$$= -\left(\int_{-\infty}^{0} H(x) f'(x)\,dx + \int_{0}^{+\infty} H(x) f'(x)\,dx\right)$$

$$= -f(x)\Big|_0^{\infty} = -(f(\infty) - f(0)) = f(0).$$

Property:

For any continuous function $g(x)$,

$$\int_{\mathbb{R}} \delta(x-x_0)\, g(x)\,dx = g(x_0).$$

Then, $\hat{\delta}(\eta) = \int e^{-i\eta x}\delta(x)\,dx = e^{-i\eta 0} = 1$

Properties of Fourier Transform:

① $\widehat{\partial_x f} = \displaystyle\int_{-\infty}^{\infty} e^{-i\eta x}\partial_x f\,dx \overset{\text{I.B.P}}{=} -\int_{-\infty}^{+\infty} (e^{-i\eta x})_x f\,dx = i\eta\int_{-\infty}^{\infty} f\cdot e^{-i\eta x}dx$

$$= i\eta\cdot\hat{f}(\eta)$$

$\Downarrow$

$$\boxed{(\partial_x f)^\wedge = i\eta\cdot\hat{f}(\eta)}$$

Application; $\begin{cases} (\partial_t + c\,\partial_x)u = 0 \\ u(x,0) = u_0(x) \end{cases}$

Take Fourier transform:

$$\partial_t \hat{u} + c i \eta \hat{u} = 0 \quad , \quad \eta \text{ fixed}$$

$$\Rightarrow \partial_t \left(e^{ci\eta t} \hat{u} \right) = 0 \Rightarrow \hat{u}(\eta, t) = e^{-ic\eta t} \hat{u}(\eta, 0)$$

$$= e^{-ic\eta t} \cdot \int_{-\infty}^{\infty} e^{-i\eta x} u_0(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-i\eta(x+ct)} u_0(x)\, dx$$

Change $z = x + ct$.
$x = z - ct$

$$\Rightarrow = \int_{-\infty}^{\infty} e^{-i\eta z} u_0(z - ct)\, dz$$

$$= \int_{-\infty}^{\infty} e^{-i\eta x} u_0(x - ct)\, dx = u_0(x-ct)^\wedge$$

② $$\boxed{f(x-a)^\wedge = e^{-ia\eta} \hat{f}}\ ;$$

Laplace Transform / Fourier Transform

• Convolution:

$$f * g(x) = \int_{\mathbb{R}} f(x-y) \cdot g(y)\, dy$$

③ $$\boxed{(f*g)^\wedge = \hat{f} \cdot \hat{g}}\ ;$$

Notice, $\hat{u}(\eta, t) = e^{-ic\eta t} \cdot \hat{u}(\eta, 0)$
$= \delta(x-ct)^\wedge \cdot \hat{u}(\eta, 0)$

for $\delta(x-ct) = \hat{\delta}(\eta) \cdot e^{-ict\eta} = e^{-ict\eta}$

$$\Rightarrow u(x,t) = \delta(x-ct) * u(x,0)$$

$$= \int \delta(x-ct-y) \cdot u(y,0)\, dy = u(x-ct, 0)$$

$$= u_0(x-ct). \ \#$$

R.K: Thus, $(e^{-i\eta ct}) = \widehat{\delta(x-ct)}$ is the Green Function of $u_t + c u_x = 0$. & $u(x,t) = \delta(x-ct) * u_0(x)$.

Recall:

Fourier Transform:

$$\delta(x) \longrightarrow 1$$

$$\delta(x-ct) \longrightarrow e^{-i\eta ct}$$

$$f \longrightarrow \hat{f}(\eta)$$

$$\hat{f'} \longrightarrow i\eta \hat{f}(\eta)$$

$$\widehat{f*g} \longrightarrow \hat{f}(\eta)\cdot\hat{g}(\eta)$$

Inverse Fourier transform:

$$\hat{f} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)\cdot e^{i\eta x}\,d\eta$$

Now,

$$\boxed{\begin{cases} g_t + c g_x = S(x,t) \\ g(x,0) = g_0(x) \end{cases}}$$

Take Fourier transform: "PDE ⇒ ODE"

$$\Longrightarrow \boxed{\begin{cases} \hat{g}_t + c i\eta \hat{g} = \hat{S}(\eta,t) \\ \hat{g}(\eta,0) = \hat{g}_0(\eta) \end{cases}}$$

Next, $e^{i\eta ct}$.

$$\Rightarrow \quad \frac{d}{dt}\left(e^{i\eta ct}\hat{g}\right) = e^{i\eta ct}\hat{S}(\eta,t)$$

Take integral from 0 to τ,

$$\int_0^\tau \frac{d}{dt}\left(e^{i\eta ct}\cdot\hat{g}\right)dt = \int_0^\tau e^{i\eta ct}\,\hat{s}(\eta,t)\,dt$$

$$\Rightarrow\quad e^{i\eta c\tau}\cdot\hat{g}(\eta,\tau) - \hat{g}_0(\eta) = \int_0^\tau e^{i\eta ct}\,\hat{s}(\eta,t)\,dt$$

$$\Rightarrow\quad \hat{g}(\eta,\tau) - e^{-i\eta\tau c}\hat{g}_0(\eta) = \int_0^\tau e^{i\eta c(t-\tau)}\,\hat{s}(\eta,t)\,dt$$

$$\delta(x-c\tau) \qquad\qquad \delta(x-c(\tau-t))$$

$$g(x,\tau) - \delta(x-c\tau)*g_0 = \int_0^\tau \delta(x-c(\tau-t))*s(x,t)\,dt$$

$$\int \delta(x-c\tau-y)\,g_0(y)\,dy$$

$$= g_0(x-c\tau)$$

final result.

$$\Rightarrow\quad \boxed{g(x,\tau) = g_0(x-c\tau) + \int_0^\tau s(x-c(\tau-t),t)\,dt}$$

i.e. $g(x,t) = g_0 * \delta(x-ct) + \int_0^t s(x,\tau) * \delta(x-c(t-\tau))\,d\tau$

Wave Equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x) \end{cases}$$

Take Fourier Transform,

$$\begin{cases} \hat{u}_{tt} - (i\eta)^2 \hat{u} = 0 \longrightarrow \hat{u}_{tt} + \eta^2 \hat{u} = 0 \\ \hat{u}(x,0) = \hat{u}_0(\eta) \quad \text{—①} \\ \hat{u}_t(\eta,0) = \hat{u}_1(\eta) \quad \text{—②} \end{cases}$$

$$\lambda^2 + \eta^2 = 0, \Rightarrow \lambda = \pm i\eta$$

$$\Downarrow$$

$$\hat{u}(\eta,t) = A \cdot e^{i\eta t} + B e^{-i\eta t}$$

Use condition ① & ② to solve A, B.

$$\begin{cases} \hat{u}_0(\eta) = A + B \\ \hat{u}_1(\eta) = i\eta A - i\eta B \iff \frac{\hat{u}_1(\eta)}{i\eta} = A - B \end{cases}$$

So, $A = \frac{1}{2}\left(\hat{u}_0(\eta) + \frac{\hat{u}_1(\eta)}{i\eta}\right)$, & $B = \frac{1}{2}\left(\hat{u}_0(\eta) - \frac{\hat{u}_1(\eta)}{i\eta}\right)$

$$\Rightarrow \hat{u}(\eta,t) = \frac{1}{2}\left(\hat{u}_0(\eta) + \frac{\hat{u}_1(\eta)}{i\eta}\right) \cdot e^{i\eta t} + \frac{1}{2}\left(\hat{u}_0(\eta) - \frac{\hat{u}_1(\eta)}{i\eta}\right) \cdot e^{-i\eta t} = \boxed{\frac{1}{2}\hat{u}_0\left(e^{i\eta t} + e^{-i\eta t}\right) + \frac{1}{2}\hat{u}_1 \frac{\left(e^{i\eta t} - e^{-i\eta t}\right)}{i\eta}}$$

$$= \frac{1}{2}\left(\widehat{u_0(x+t)} + \widehat{u_0(x-t)}\right) + \frac{1}{2}\left[\int u_1(x+t) - u_1(x-t)\, dx\right]^{\wedge}$$

So, $u(x,t) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t) \right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi$.

For Inhomogeneous,

$$\boxed{\begin{cases} u_{tt} - u_{xx} = S(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}} \Rightarrow \begin{cases} \hat{u}_{tt} + \eta^2 \hat{u} = \hat{S}(\eta,t) \\ \hat{u}(\eta,0) = 0 \\ \hat{u}_t(\eta,0) = 0 . \end{cases}$$

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u_t(0,x) = \delta(x) \\ u(0,x) = 0 \end{cases} \xrightarrow{\text{Fourier Transform}}$$

Want to solve $\hat{u}(\eta,t)$:

Introduce $G(z,\eta)$, $z \in (0,t)$ s.t. $\begin{cases} G_z(t,\eta) = 1 \\ G(t,\eta) = 0 \\ (-\partial_z)^2 G + \eta^2 G = 0 \quad \text{— } (*) \end{cases}$

"Backward equation"

"Green function"

$$\int_0^t G(z,\eta)\cdot\left[\hat{u}_{zz}(\eta,z) + \eta^2 \hat{u}(\eta,z) \right] dz = \int_0^t G(z,\eta)\, \hat{S}(\eta,z)\, dz$$

$$G(z,\eta)\,\hat{u}_z \Big|_0^t - \int_0^t \hat{u}_z \cdot G_z(z,\eta)\, dz$$

$\Downarrow$ 0

$$\Downarrow$$

$$\text{"}-\left(G_z \hat{u}\Big|_0^t - \int G_{zz}\, \hat{u}\, dz \right)\text{"} +$$

$$= -G_z \hat{u}\Big|_t + \int_0^t G_{zz}\, \hat{u}\, dz +$$

$$= -\hat{u}(\eta,t) + \int_0^t G_{zz}\, \hat{u}\, dz + \int_0^t G(z,\eta)\, \eta^2\, \hat{u}(\eta,z)\, dz$$

$$= -\hat{u}(\eta,t) + \int_0^t \underbrace{(G_{zz} + \eta^2 G)}_{\overset{\Downarrow}{0} \;\Leftarrow\; \text{by } (*)} \hat{u}\, dz$$

$$\Rightarrow \hat{u}(\eta,t) = -\int_0^t G(z,\eta)\cdot \hat{S}(z,\eta)\,dz.$$

W.T.S get $G(z,\eta)$:

Let $\tilde{G}(z,\eta) = G(t+z,\eta)$.

Then,

$$\begin{cases} \tilde{G}_{zz} + \eta^2 \tilde{G} = 0 \\ \tilde{G}_z(0,\eta) = 1 \\ \tilde{G}(0,\eta) = 0 \end{cases}$$

$$\Rightarrow r^2 + \eta^2 = 0 \quad \text{i.e } r = \pm i\eta$$

$$\Rightarrow \tilde{G}(z,\eta) = A\cdot e^{i\eta z} + B\cdot e^{-i\eta z}$$

$$\Rightarrow A = -B \qquad \text{by initial value}$$

$$\Rightarrow \tilde{G}_z(0,\eta) = i\eta A + i\eta A = 1 \quad \text{implying} \quad \underline{A = \frac{1}{2i\eta}}$$

$$\Rightarrow \tilde{G}(z,\eta) = \frac{1}{2i\eta}\left(e^{i\eta z} - e^{-i\eta z}\right)$$

$\uparrow$ $\delta(x+t)$ $\quad$ $\uparrow$ $\delta(x-t)$. $\quad$ F.T.

$$\Rightarrow G(z,\eta) = \frac{1}{2i\eta}\left(e^{i\eta(z-t)} - e^{-i\eta(z-t)}\right)$$

$$= \frac{1}{2i\eta}\left(e^{-i\eta(t-z)} - e^{i\eta(t-z)}\right)$$

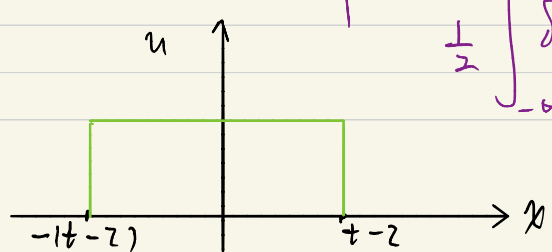
$$\Rightarrow \hat{u}(t,\eta) = \int_0^t \underline{\frac{1}{2i\eta}\left(e^{i\eta(t-z)} - e^{-i\eta(t-z)}\right)}\cdot \hat{S}(z,\eta)\,dz$$

$\uparrow$

$$\frac{1}{2}\int_{-\infty}^{x} \delta(y+(t-z)) - \delta(y-(t-z))\,dy$$

$$\Downarrow$$

$$\chi_{[-(t-z),\, t-z]}(x) * S(z,x)$$



$$\Rightarrow \hat{u}(\eta,t) = \int_0^t G(\eta,\tau)\,\hat{S}(\eta,\tau)\,d\tau$$

$$G = \frac{e^{i(t-\tau)\eta} - e^{-i\eta(t-\tau)}}{2i\eta}$$ solved by privious step.

$$\boxed{u(x,t) = \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \frac{S(\xi,\tau)\,d\xi}{2}\, d\tau}$$

Heat equation:

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t>0 \\ u(x,0) = u_0(x) \end{cases}$$

$$\begin{cases} \hat{u}_t + \eta^2 \hat{u} = 0 \\ \hat{u}(x,0) = \hat{u}_0(\eta) \end{cases} \Longrightarrow \frac{d}{dt}\left(e^{\eta^2 t}\cdot \hat{u}\right) = 0$$

$$\Downarrow$$

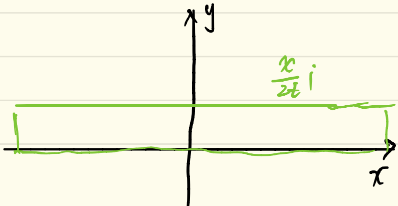
$$e^{\eta^2 t}\cdot \hat{u}(\eta,t) = \hat{u}(\eta,0)$$

$$\Rightarrow \hat{u}(\eta,t) = e^{-\eta^2 t}\, \underbrace{\hat{u}(\eta,0)}_{\text{known}}$$

Look for its inverse.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\eta^2 t + i\eta x}\, d\eta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{e^{-\eta^2 t + i\eta x + \frac{x^2}{4t} - \frac{x^2}{4t}}}_{e^{-t(\eta - \frac{ix}{2t})^2}\cdot e^{-\frac{x^2}{4t}}}\, d\eta = \frac{e^{-x^2/4t}}{2\pi}\cdot \int_{-\infty}^{\infty} e^{-t(\eta - \frac{ix}{2t})^2}\, d\eta$$

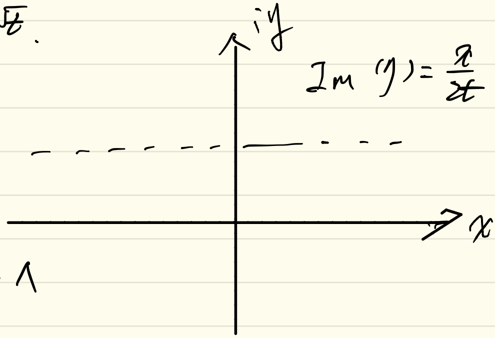


By Complex analysis

$$= \frac{e^{-\frac{x^2}{4t}}}{2\pi} \cdot \int_{Im(y)=\frac{x}{2t}} e^{-t(y-\frac{ix}{2t})^2}\, dy = \frac{e^{-x^2/4t}}{2\pi} \int_{-\infty}^{\infty} e^{-tv^2} \frac{dv\sqrt{t}}{\sqrt{t}}, \qquad v = y - \frac{ix}{2t}$$

$$= \frac{e^{-x^2/4t}}{2\pi\sqrt{t}} \cdot \int_{-\infty}^{\infty} e^{-w^2}\, dw, \qquad w = v\sqrt{t}.$$

$$= \frac{\sqrt{\pi}}{2\pi\sqrt{t}} \cdot e^{-x^2/4t}$$



So, $\hat{u}(x,t) = \left[\frac{e^{-x^2/4t}}{\sqrt{4\pi t}} * u(x,0)\right]^{\wedge}$

$$u(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} * u(x,0) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} \cdot u(y,0)\, dy.$$

$$\boxed{\text{Heat kernel: } k(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}} \qquad \underline{\hat{k}(y,t) = e^{-y^2 t}.}$$

Duhamel's Principle:

$$(*) \begin{cases} u_t - u_{xx} = S(x,t) \\ u(x,0) = 0 \end{cases} \xRightarrow{\text{By Fourier Transform}} \begin{cases} \hat{u}_t + y^2 \hat{u} = \hat{S}(y,t) \quad \text{— ☺} \\ \hat{u}(y,0) = 0 \end{cases}$$

Integrate ☺ by $e^{y^2 t}$:

$$\Rightarrow \frac{d}{dt}(e^{y^2 t} \cdot \hat{u}) = \hat{S}(y,t) \cdot e^{y^2 t}$$

Change "t" to "τ",

$$\int_0^t \left[\frac{d}{d\tau}(e^{y^2\tau} \cdot \hat{u}) = \hat{S}(y,\tau)\, e^{y^2\tau}\right] d\tau$$

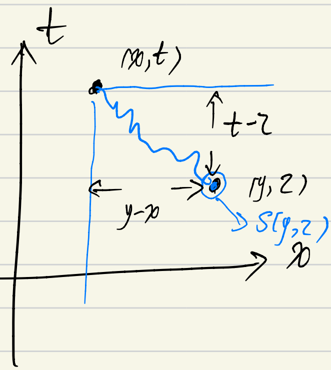
$$\Rightarrow \quad e^{\eta^2 t}\hat{u}(\eta,t) = \int_0^t \hat{S}(\eta,\tau)\, e^{\eta^2 \tau}\, d\tau$$

$$\Rightarrow \quad \hat{u}(\eta,t) = \int_0^t \underbrace{e^{\eta^2(\tau-t)}}_{k(x,\,t-\tau)} \cdot \hat{S}(\eta,\tau)\, d\tau$$

$$\Rightarrow \quad \boxed{u(x,t) \equiv \int_0^t k(x,\, t-\tau) * S(x,\tau)\, d\tau}$$

i.e.

$$\boxed{u(x,t) = \int_0^t \int_{-\infty}^{+\infty} k(x-y,\, t-\tau) \cdot S(y,\tau)\, dy\, d\tau}$$



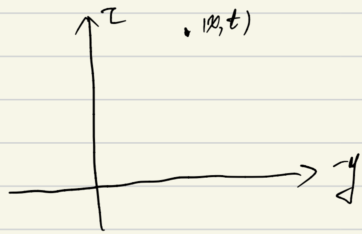
Now, For general,

(Forward Eq.) $\begin{cases} u_t(x,t) = u_{xx} + S(x,t) \\ u(x,0) = u_0(x) \end{cases} \Big\} \longrightarrow$

Then, $u(x,t) = k(x,t) * u_0(x) + \int_0^t k(x,t-\tau) * S(x,\tau)\, d\tau$

(ii)

Introduce a function $G(y,\tau)$



s.t.

Backward Eq. $\begin{cases} ① \ G(y,t) = \delta(x-y) \\ ② \ [-\partial_\tau - (-\partial_y)^2]\, G(y,\tau) = 0 \end{cases}$

For $\int_0^t \int_{-\infty}^{\infty} [u_\tau(y,\tau) - u_{yy}(y,\tau) - S(y,\tau) = 0\,]\, G(y,\tau)\, dy\, d\tau$

$$\Rightarrow \int_{-\infty}^{+\infty} G(y,\tau)\, U(y,\tau)\, dy \Big|_{\tau=0}^{\tau=t} + \int_0^t \int_{-\infty}^{+\infty} \left((-\partial_\tau - (-\partial_y)^2)\, G\right) U(y,\tau)$$

$$- G(y,\tau)\, S(y,\tau)\, dy\, d\tau = 0$$

$$\Rightarrow \quad u(x,t) - \int_{-\infty}^{\infty} G(y,0)\,U(y,0)\,dy + \int_0^t\int_{-\infty}^{\infty} G(y,z)\,S(y,z)\,dy\,dz$$

$$\Rightarrow \quad G(y,z) = k(x-y,\ t-z).$$

"Hard" $\boxed{\text{A nonlinear problem:}}$

$$\begin{cases} u_t + u u_x = u_{xx}. \\ |u(x,0)| \le \varepsilon \cdot \dfrac{e^{-\frac{x^2}{4}}}{\sqrt{16\pi}}, \quad u(x,0) = U(x). \end{cases}$$

How to construct a sol. ?

Observe: $u u_x$ is $O(\varepsilon^2)$ & much smaller then other terms.

Thus, write

$$u_t - u_{xx} = -\left(\frac{u^2}{2}\right)_x$$

By Duhamel's principle,

$$u(x,t) = k(x,t) * U(x) + \underbrace{\int_0^t k(x,t-z) * \left(-\left(\frac{u^2}{2}\right)_x\right) dz}_{-\int_0^t k(x,t-z) * \left[\frac{u^2(x,z)}{2}\right]_x dz}$$

So, $u(x,t) = k(x,t) * U(x) \underbrace{- \int_0^t k(x,t-z) * \left[\frac{u^2(x,z)}{2}\right]_x dz}$

$$\Downarrow$$

$$\underbrace{-\int_0^t \int_{-\infty}^{\infty} k(x-y,t-z) \left(\frac{u^2(y,z)}{2}\right)_y dy\,dz}$$

$$\Downarrow$$

$$+\int_0^t\int_{-\infty}^{\infty} k_y(x-y,t-z) \cdot \frac{u^2(y,z)}{2}\,dy\,dz$$

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t) \cdot U(y)\,dy + \int_0^t\int_{-\infty}^{\infty} k_y(x-y,t-z) \cdot \frac{u^2(y,z)}{2}\,dy\,dz$$

Construct iteration:

$$u_{(k)}(x,t) = \int_{-\infty}^{\infty} k(x-y,t) \cdot U(y)\,dy + \int_0^t\int_{-\infty}^{\infty} k_y(x-y,t-z)\, \frac{u^2_{(k-1)}(y,z)}{2}\,dy\,dz,$$

$k \ge 1$

Initial u_0: $\boxed{u_0(x,t) = \int_{-\infty}^{\infty} k(x-y,t)\, U(y)\, dy = k * U(x)}$

$\Rightarrow$ $|u_0(x,t)| \le \varepsilon \int_{-\infty}^{\infty} k(x-y,t)\, k(y,1)\, dy = \varepsilon\, k(x, t+1)$

Assume $|u_k(x,t)| \le 2\varepsilon\, k(x/2, t+1)$.

By introduction, this is true for $k \in \mathbb{N}$.

Define a norm,

$$|||f||| = \sup_{(x,t)} \frac{f(x,t)}{k(\frac{x}{2}, t+1)} \Rightarrow |||u_0||| \le \sup_{(x,t)} \frac{|k(x,t+1)|\varepsilon}{k(x/2, t+1)}$$

$$\le \varepsilon$$

By constructing,

$$|u_k - u_{k-1}| \le \int_0^t \int_{-\infty}^{\infty} |k_y(x-y, t-\tau)|\, \Big[\frac{(u_{k-1} - u_{k-2})}{\cancel{2}}\Big] \cdot \big|[\underbrace{u_{k-1} + U_{k-2}}]\big|\, dx\, d\tau$$

$4\varepsilon k(\frac{y}{2}, \tau+1)$ $\quad$ $|||u_{k-1} + u_{k-2}|||$.

$$\le 4\varepsilon\, |||u_{k-1} - u_{k-2}||| \cdot \int_0^t \int_{-\infty}^{+\infty} k_y(x-y, t-\tau) \cdot k(y/2, \tau+1) \cdot k^2(y/2, \tau+1)\, dy\, d\tau$$

$$|k_y(x-y, t-\tau)| = \Big|\frac{2(x-y)}{2\cdot(t-\tau)}\, k(x-y, t-\tau)\Big|$$
$$\le \frac{C}{\sqrt{t-\tau}} \cdot k(\frac{x-y}{2}, t-\tau).$$

$\Rightarrow$ $k_y^2(y/2, \tau+1) \le C \cdot k(\frac{y}{2}, \tau+1)/\sqrt{2}$

Eventually, $\le 4\varepsilon\, |||u_{k-1} - u_{k-2}||| \cdot \int_0^t k(\frac{x}{2}, t+1) \cdot \frac{1}{\sqrt{\tau+1}} \frac{1}{\sqrt{t-\tau}}\, d\tau \cdot C$

$\le 4\varepsilon C_1 \cdot |||u_{k-1} - u_{k-2}||| \cdot k(\frac{x}{2}, 1+t)$

$\Rightarrow$ $|||u_k - u_{k-1}||| \le 4\varepsilon C_1 \cdot |||u_{k-1} - u_{k-2}|||$

Observe:

$\mathcal{F}: \{|||u||| \le 2\varepsilon\} \longrightarrow \{|||u||| \le 2\varepsilon\}$: complete metric space.

- $\mathcal{F}(u_k) = u_{k+1}$

- $\mathcal{F}$ is a contraction:

$$|||\mathcal{F}(u_{k-1}) - \mathcal{F}(u_{k-2})||| = |||u_k - u_{k-1}|||$$

By fixed point Thm,

$$\leq 4\varepsilon C_1 \cdot |||u_{k-1} - u_{k-2}|||$$

$\Longrightarrow$ $\exists!$ u s.t.

$$\mathcal{F}(u) = u.$$

Q.E.D.

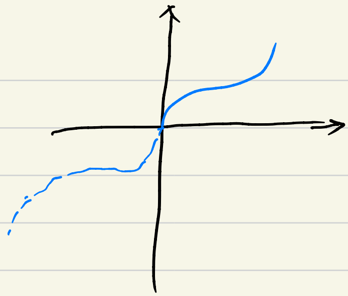
Recall:

$\boxed{\text{Case 1}}$: $\begin{cases} u_{tt} = u_{xx}, & x>0 \\ u(0,t)=0, & u(x,0)=u_0(x), \quad u_t(x,0)=u_1 \end{cases}$

By odd extension,

$f(x), \quad x>0$

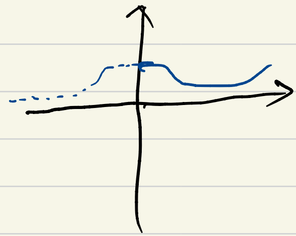
$$f_{odd}(x) = \begin{cases} f & \text{if } x>0 \\ -f(-x), & \text{if } x<0 \end{cases}$$



$\boxed{\text{Case 2}}$: $\begin{cases} u_{tt} = u_{xx}, & x>0 \\ u_x(0,t)=0, & u(x,0)=u_0(x), \quad u_t(x,0)=u_1 \end{cases}$

By even extension:

$$f^{even}(x) = \begin{cases} f(x), & \text{if } x>0 \\ f(-x), & \text{if } x<0 \end{cases}$$



- What if for mixed bdd?

$$\begin{cases} u_{tt} = u_{xx}, & x>0 \\ \underline{u_x(0,t) = a\cdot u(0,t)}, & u(x,0)=u_0(x), \quad u_t(x,0) = v_1(x) \end{cases}$$

Now, there is a new method, which can solve the problem via $\boxed{\text{Laplace Transform}}$.

☺ $\begin{cases} u_t = u_{xx}, & x\in\mathbb{R}, t>0 \\ u(x,0) = \delta(x) \end{cases} \Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}$

$\boxed{\underline{\text{Laplace Transform}}: \quad \mathcal{L}u(x,s) = \int_0^\infty e^{-st}\cdot u(x,t)\,dt, \quad Re(s)\geq 0.}$

Properties:

① $\mathcal{L}u_t(x,s) = \int_0^\infty e^{-st}\cdot u_t(x,t)\,dt = e^{-st}\cdot u(x,t)\Big|_0^\infty + s\cdot\int_0^\infty e^{-st}u(x,t)dt$

$$\Rightarrow \boxed{\mathcal{L}u_t(x,s) = -u(x,0) + s\cdot\mathcal{L}u(x,s)}$$

For ☺, $\begin{cases} \mathcal{L}u_t = \partial_{xx}^2 \mathcal{L}u \\ u(x,0) = \delta(x) \end{cases} \Rightarrow \underline{s\mathcal{L}u - \delta(x) = \partial_{xx}^2\mathcal{L}u}$. This is ODE.

So, $\mathcal{L}u = \begin{cases} A_+\cdot e^{-\sqrt{s}x} + B_+\cdot e^{\sqrt{s}x}, & x>0 \\ A_-\cdot e^{-\sqrt{s}x} + B_- e^{\sqrt{s}x}, & x<0 \end{cases} \Rightarrow$ By contin. at $x=0$, $A_+ = B_+$

So, $\mathcal{L} u_x = \begin{cases} -\sqrt{s} A_+ e^{-\sqrt{s} x}, & \text{if } x>0 \\ \sqrt{s} A_+ e^{\sqrt{s} x}, & \text{if } x<0 \end{cases}$ $\Rightarrow$ The size of the jump is: $-2\sqrt{s} \cdot A_+$

$\Rightarrow -2\sqrt{s} \cdot A_+ = -1$, i.e. $A_+ = \frac{1}{2\sqrt{s}}$

Finally, $\mathcal{L} u = \begin{cases} \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s} x}, & x>0 \\ \frac{1}{2\sqrt{s}} \cdot e^{\sqrt{s} x}, & x<0 \end{cases}$ (*)

R.K.: Using this to solve lots of eq-s.

$\Leftrightarrow \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|x|}$, $\Rightarrow u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$.

Eg 1.: $\begin{cases} u_t = u_{xx}, & x>0 \\ u(0,t)=0, & u(x,0) = u_0(x). \end{cases}$

By odd extension, u^{odd}; $\begin{cases} u^{odd}_t = u^{odd}_{xx}, & x \in \mathbb{R}' \\ u(x,0) = u_0^{odd}(x) \end{cases}$

By Duhamel's principle:

$$u^{odd}(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * u_0^{odd}(x)$$

"Green function of Eg 1:" $g(x,t)$.

$$\Rightarrow u^{odd}(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \cdot u_0(y)\,dy = \int_{-\infty}^{0} + \int_0^{+\infty} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} \cdot u_0(y)\,dy = \int_0^{\infty} \boxed{\left(\frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} - \frac{e^{-\frac{(x+y)^2}{4t}}}{\sqrt{4\pi t}}\right)} u_0(y)\,dy$$

So, $\Rightarrow \begin{cases} g_t = g_{xx}, & x>0 \\ g(0,t,y) = 0 \\ g(x,0,y) = \delta(x-y) \end{cases}$ Applying Lap. Transform, $\Rightarrow \begin{cases} s\mathcal{L}g - \delta(x-y) = \partial_x^2 \mathcal{L} g \\ \mathcal{L} g(0,s,y) = 0 \end{cases}$

$\Rightarrow \mathcal{L} g - \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}\cdot|x-y|}$ defined $\alpha(x,s)$

$\Rightarrow \begin{cases} s\alpha(x,s) - \alpha_{xx}(x,s) = 0 \\ \alpha(0,s) = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|y|} \end{cases}$ & $\alpha(x,s) = A_+ e^{-\sqrt{s} x}$, $x>0$

$\rightarrow A_+ = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|y|}$

$\Rightarrow \alpha(x,s) = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}(x+y)}$, $\Rightarrow \mathcal{L} g = \frac{1}{2\sqrt{s}} \left(e^{-\sqrt{s}|x-y|} - e^{-\sqrt{s}|x+y|}\right)$

$$\Rightarrow g(x,t;y) = \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} - \frac{e^{-\frac{(x+y)^2}{4t}}}{\sqrt{4\pi t}}. \quad \#$$

verify: $\mathcal{L}u(x,s) = \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}\cdot|x|}$,

$u(x,t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}$

$$\mathcal{L}u = \int_0^{+\infty} \frac{e^{-st} \cdot e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\, dt ,$$

Let $\tilde{t} = \sqrt{t}$

$$= \int_0^{\infty} \frac{1}{\cancel{2}\sqrt{\pi}\,\cancel{\tilde{t}}} \cdot e^{-s\tilde{t}^2 - \frac{x^2}{4\tilde{t}^2}} \cdot \cancel{2}\cancel{\tilde{t}}\, d\tilde{t}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \underbrace{e^{-s\tilde{t}^2 - \frac{x^2}{4\tilde{t}^2}}}\, d\tilde{t}$$

$$= \frac{e^{-\sqrt{s}|x|}}{\sqrt{\pi}} \cdot \int_0^{\infty} e^{-(\sqrt{s}\,\tilde{t} + \frac{|x|}{2\tilde{t}})^2}\, d\tilde{t} , \quad \text{Let } y = \sqrt{s}\,\tilde{t} + \frac{|x|}{2\tilde{t}}$$

Note $\boxed{\int_0^{\infty} e^{-(ax - \frac{b}{x})^2}\, dx = \frac{1}{a} \cdot \int_0^{\infty} e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2a}}$

$$\Rightarrow \mathcal{L}u = e^{-\sqrt{s}\cdot|x|} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2\sqrt{s}}$$

$$= \frac{e^{-\sqrt{s}\cdot|x|}}{2\sqrt{s}}$$

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Solution for the following HW:

Homework: PDEs.

A0187036X
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Problem:

$$\begin{cases} u_t = u_{xx}, & x>0 \\ u(0,t)+u_x(0,t)=0 \\ u(x,0)=\delta(x-y), & y>0. \end{cases}$$

Solution: By laplace transform: $\mathcal{L}u = \int_0^\infty u(x,t)\cdot e^{-st}\,dt$

Thus, $\mathcal{L}u_t(x,s) = \int_0^\infty u_t \cdot e^{-st}\,dt \overset{I.B.P}{=} s\cdot\mathcal{L}u - u(x,0) = s\cdot\mathcal{L}u - \delta(x-y)$.

And we get

$$\underline{s\cdot\mathcal{L}u - \delta(x-y) = \partial_x^2\mathcal{L}u}$$

The boundary condition is

$$\underline{\mathcal{L}u(0,t) + \partial_x\mathcal{L}u(0,t) = 0} \quad \text{— B.C.}$$

Remember, previously in class, we already solve:

$$\begin{cases} \tilde u_t = \tilde u_{xx}, & x\in\mathbb{R}, t>0 \\ \tilde u(x,0) = \delta(x) \end{cases}$$

The solution is $\tilde u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$.

Consider $\tilde u(x-y,t) = \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}$,

It satisfies the Laplace transform:

$$\mathcal{L}\tilde u_t(x-y,s) = s\cdot\mathcal{L}\tilde u(x-y,s) - \delta(x-y) = \partial_x^2\hat{\tilde u}(x-y,s)$$

Now, take $H(x,t) = u(x,t) - \tilde u(x-y,t)$

Observe:

$$s\,\mathcal{L}H = (\mathcal{L}u - \mathcal{L}\tilde u(x-y,s))\cdot s \quad \&$$

$$\partial_x^2\mathcal{L}H = \partial_x^2\mathcal{L}u - \partial_x^2\mathcal{L}\tilde u(x-y,s)$$

Thus, $\underline{s\mathcal{L}H = \partial_x^2\mathcal{L}H}$ for $s\mathcal{L}u = \delta(x-y) + \partial_x^2\mathcal{L}u$
& $s\mathcal{L}\tilde u(x-y) = \partial_x^2\mathcal{L}\tilde u(x-y) + \delta(x-y)$

So, $\mathcal{L}H = \alpha\cdot e^{-\sqrt{s}x}$, α — to be determined.

Then, $\mathcal{L}u = \mathcal{L}H + \mathcal{L}\tilde u(x-y) = \alpha\cdot e^{-\sqrt{s}x} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}$

By B.C.,

$$\alpha + \frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}} + (-\sqrt{s}\,\alpha) + (\sqrt{s})\cdot\frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}} = 0$$

$$\Rightarrow \alpha = -\frac{(1+\sqrt{s})}{1-\sqrt{s}}\cdot\frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}}$$

So,

$$\mathcal{L}u = -\frac{(1+\sqrt{s})}{1-\sqrt{s}}\cdot\frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}}\cdot e^{-\sqrt{s}x} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}, \quad y>0.$$

$$\mathcal{L}u = -\left(-1+\frac{2}{1-s}+\frac{2}{1-s}\sqrt{s}\right)\cdot e^{-\sqrt{s}(y+x)} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}, \quad y>0,\ x>0$$

$$= \left(1-\frac{2}{1-s}+\frac{1}{1-s}\sqrt{s}\right)\cdot e^{-\sqrt{s}|x+y|} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}.$$

observe: $k(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$, $\cdot\ \mathcal{L}k(x,s) = \frac{1}{2\sqrt{s}}\cdot e^{-\sqrt{s}|x|}$

$$\cdot\ 2\sqrt{s}\cdot\frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}} = -2\partial_x\left(\frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}}\right)$$

$$= -2\mathcal{L}(k_x(x+y,t))$$

$$\cdot\ -\frac{2}{1-s}\cdot e^{-\sqrt{s}|x+y|} = -2\mathcal{L}(e^t)\cdot(-2)\cdot\mathcal{L}k_x(x+y,t)$$

$$= 4\mathcal{L}(e^t)\cdot\mathcal{L}k_x(x+y,t) \quad \text{for } \mathcal{L}f\cdot\mathcal{L}g = \mathcal{L}(f*g)$$

$$= 4\mathcal{L}(e^t * k_x(x+y,t))$$

$$\cdot\ \frac{2}{1-s}\sqrt{s}\cdot e^{-\sqrt{s}|x+y|} = \frac{4s}{1-s}\cdot\frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}}$$

$$= \left(-4+\frac{4}{1-s}\right)\cdot\mathcal{L}k(x+y,t)$$

$$= -4\mathcal{L}k(x+y,t) + 4\mathcal{L}(e^t)\cdot\mathcal{L}k(x+y,t)$$

$$= -4\mathcal{L}k(x+y,t) + 4\mathcal{L}(e^t * k(x+y,t)).$$

$$= -4\mathcal{L}k(x+y,t) + 4\mathcal{L}(e^t * k(x+y,t)).$$

So,
$$u(x,t) = -2k_x(x+y,t) + 4e^t * k_x(x+y,t) + k(x-y,t), \text{ where}$$
$$-4k(x+y,t) + 4e^t * k(x+y,t). \qquad k(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$$

$\square$.

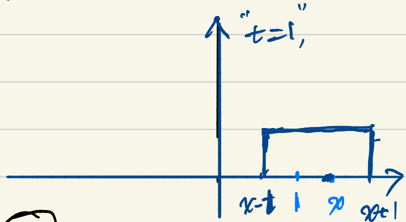
HW: Solve
$$\begin{cases} g_t - g_{xx} = 0, & x>0 \\ g_x(0,t) + g(0,t) = 0 \\ g(x,0;y) = \delta(x-y), & y>0 \end{cases}$$

Class 6.

$$\begin{cases} u_{tt} - u_{xx} = 0, & x>0, \ t>0 \\ u(x,0)=0, \quad u_t(x,0) = \delta(x-y), \quad u(0,t)=0, & y>0 \end{cases}$$

Recall for a whole space problem:
$$\begin{cases} u_{tt} - u_{xx} = 0, & x\in\mathbb{R}, \ t>0 \\ u_t(x,0) = u_1(x), & u(x,0) = u_0(x) \end{cases}$$
$$u(x,t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\,d\xi$$

If $u_1(x) = \delta(x)$, & $u_0(x)=0$, then
$$u(x,t) = \begin{cases} 1, & \text{if } |x|<t \\ 0, & \text{if } |x|>t \end{cases}$$



Consider
$$\begin{cases} u_{tt} - u_{xx} = 0, & x\in\mathbb{R}, \ t>0 \\ u_t(x,0) = \delta(x), & u(x,0)=0 \end{cases}$$ —— ☺

Apply $\underbrace{\text{Laplace Transform}}_{(s>0)}$ to ☺:
$$\underbrace{s^2\mathcal{L}u - u_t(x,0) - s\,u(x,0)}_{\mathcal{L}u_{tt}} - \partial_x^2\mathcal{L}u = 0$$

$$\Rightarrow s^2\mathcal{L}u - \partial_x^2\mathcal{L}u = \delta(x) \quad -(*)$$

$$\Rightarrow \mathcal{L}u = \begin{cases} \cancel{A_+ e^{sx}} + B_+ e^{-sx}, & x>0 \\ A_- e^{sx} + \cancel{B_- e^{-sx}}, & x<0 \end{cases}$$

By contin. at $x=0$
$\Rightarrow A_- = B_+$

$$\Rightarrow \mathcal{L}u = \begin{cases} B_+ e^{-sx}, & x>0 \\ B_+ e^{sx}, & x<0 \end{cases}$$

And $\partial_x \mathcal{L}u = \begin{cases} -s B_+ e^{-sx}, & x>0 \\ s \cdot B_+ \cdot e^{sx}, & x<0 \end{cases}$,

By $(*)$, $\partial_x \mathcal{L} u = -H(x) \Rightarrow 2sB_+ = 1$

$$\Rightarrow \mathcal{L}u = \begin{cases} \frac{1}{2s} e^{-sx}, & x>0 \\ \frac{1}{2s} \cdot e^{sx}, & x<0 \end{cases} \qquad \text{i.e } \underline{\mathcal{L}u = \frac{e^{-s|x|}}{2s}}$$

Now,

$$\begin{cases} u_{tt} - u_{xx} = 0, & x>0, \ t>0 \\ u(x,0)=0, \quad u_t(x,0) = \delta(x-y), & y>0 \\ u(0,t) = 0. \end{cases}$$

Laplace Transform:

$$\begin{cases} s^2 \mathcal{L}u - \partial_x^2 \mathcal{L}u = \delta(x-y), & x>0, \ y>0 \\ \mathcal{L}u(0,s) = 0 \end{cases}$$

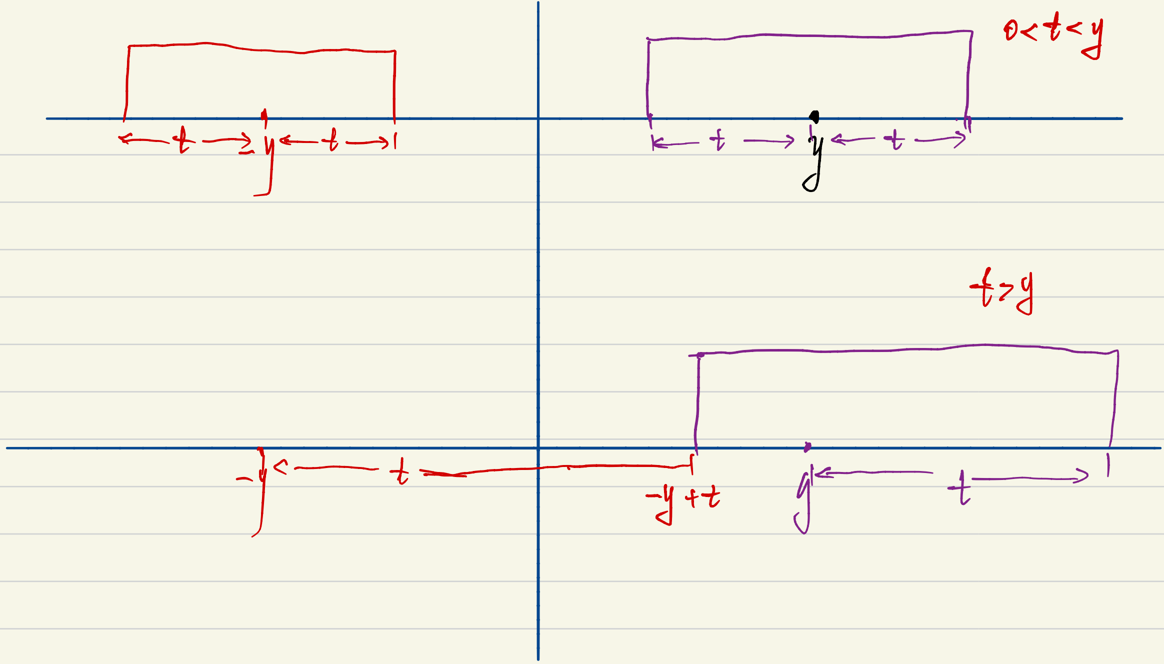
$$\Rightarrow s^2 \underbrace{\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right)}_{☺} - \partial_x^2 \underbrace{\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right)} = 0, \quad x>0$$

$$\Rightarrow ☺ = A \cdot e^{-sx} \quad \text{for } x>0$$

$$\Rightarrow ☺\Big|_{x=0} = -\frac{e^{-s|y|}}{2s} = A \quad \text{i.e. } A = -\frac{e^{-sy}}{2s} \quad \text{for } y>0$$

$$\Rightarrow ☺ = -\frac{e^{-s(y+x)}}{2s}$$

$$\Rightarrow \mathcal{L}u = \frac{e^{-s|x-y|}}{2s} - \frac{e^{-s(x+y)}}{2s}, \quad x>0, \ y>0$$



Now,
$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x>0, \quad t>0 \\ u(x,0) = 0, \quad u_t(x,0) = \delta(x-y), \quad y>0 \\ u_x(0,t) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} s^2 \mathcal{L}u - \partial_x^2 \mathcal{L}u = \delta(x-y) \\ \underline{\partial_x \mathcal{L}u(0,s) = 0} \end{cases}$$

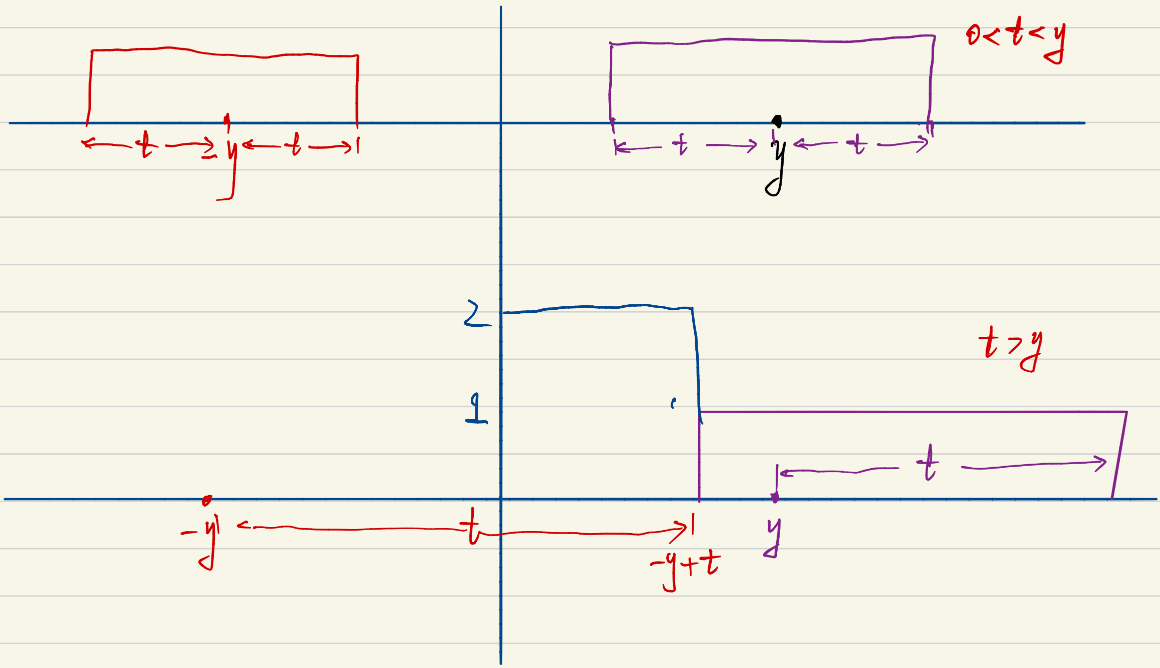
$$\Rightarrow s^2 \underbrace{\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right)}_{\text{☺}} - \partial_x^2 \underbrace{\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right)} = 0, \quad x>0$$

Then, $\text{☺} = A \cdot e^{-sx}$

By B.C. $\quad \partial_x \text{☺}\Big|_{x=0} = \partial_x\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right) = \partial_x A \cdot e^{-sx}$

$$\Rightarrow \quad -\frac{e^{-sy}}{2} = -sA \qquad \text{i.e. } A = \frac{e^{-sy}}{2s}$$

$$\Rightarrow \quad \mathcal{L}u = \frac{e^{-s|x-y|}}{2s} + \frac{e^{-s(x+y)}}{2s}, \quad x>0,\ y>0$$



Functional Analysis:

$$H^1(R) = \{ f \mid \|f\|_2 + \|f_x\|_2 < \infty \}$$

- $\|f\|_{H^1} = \sqrt{\|f\|_2^2 + \|f_x\|_2^2} = \left(\int_{\mathbb{R}} |f|^2 + |f_x|^2 \, dx \right)^{1/2}$
- $\|f\|_\infty = \operatorname{ess\,sup}_{x \in R} |f(x)|$

$\forall f$,

$$f^2(x) = \left| \int_{-\infty}^{x} \frac{d}{dx} (f^2(x)) \, dx \right| = \left| \int_{-\infty}^{x} 2 f f_x \, dx \right|$$

$$\leq 2 \int_{-\infty}^{x} |f| \cdot |f_x| \, dx \leq \int_{-\infty}^{x} |f|^2 + |f_x|^2 \, dx$$

$$\leq (\|f\|_{H^1})^2$$

$$\Rightarrow \quad \|f\|_\infty \leq \|f\|_{H^1} \quad \#.$$

Sobolev's space: $f(\vec{x})$: $\vec{x} \in \mathbb{R}^n$

$$\hat{f}(\vec{y}) = \int_{\mathbb{R}^n} f(\vec{x}) \cdot e^{-i \vec{y} \vec{x}} \, d\vec{x} \quad \text{— Fourier transform}$$

$$\|f\|_{H^k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\vec{y}|^2)^k \cdot |\hat{f}(\vec{y})|^2 \, dy \right)^{1/2}$$

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\vec{y}) \cdot e^{i \vec{y} \vec{x}} \, d\vec{y}$$

Suppose $f \in H^k(\mathbb{R}^n)$, with $k > \frac{n}{2}$. Then,

$$|f(x)| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \vec{y} \vec{x}} \cdot \hat{f}(\vec{y}) \, d\vec{y} \right|$$

$$= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \vec{y} \cdot \vec{x}} \cdot \hat{f}(\vec{y}) \cdot \frac{(1 + \vec{y}^2)^{k/2}}{(1 + \vec{y}^2)^{k/2}} \, d\vec{y} \right|$$

Schwarz ineq.

$$\leq \frac{1}{(2\pi)^n} \cdot \left(\int_{\mathbb{R}^n} |\hat{f}(\vec{y}) \cdot (1 + \vec{y}^2)^{k/2}|^2 \, d\vec{y} \right)^{1/2} \cdot \left(\int_{\mathbb{R}^n} \frac{d\vec{y}}{(1 + \vec{y}^2)^k} \right)^{1/2}$$

$$\leq \frac{1}{(2\pi)^n} \cdot \|f\|_{H^k} \cdot \left(\int_0^\infty \int_{S^{n-1}} \frac{r^{n-1}}{(1 + r^2)^k} \, d\Omega \right)^{1/2}$$

Integrable!

Class 7: (Telegraph Equation)

Caution: In previous class, we can construct the sol. by Fourier Transform, but in some cases, there exist things can not be transformed by inverse Fourier transform !!!

Telegraph Eq.

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0, & x \in \mathbb{R},\ t > 0, \quad \text{Cauchy Problem} \\ u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{cases}$$

— 

Take Fourier Transform:

$$\begin{cases} \hat{u}_{tt} + \eta^2 \hat{u} + \hat{u}_t = 0 \\ \hat{u}(\eta,0) = 0 \\ \hat{u}_t(\eta,0) = 1 \end{cases} \qquad \text{"ODE"}$$

Fix η, solve "ODE": Characteristic eq. $\xi^2 + \xi + \eta^2 = 0$

$\Rightarrow \xi_\pm = \dfrac{-1 \pm \sqrt{1-4\eta^2}}{2}$

$\Rightarrow \hat{u}(\eta,t) = (A e^{\xi_+ t} + B e^{\xi_- t})$

$\Rightarrow B = -A$ for $\hat{u}(\eta,0) = 0$

& $\hat{u}_t(\eta,t) = A\xi_+ e^{\xi_+ t} - A\xi_- e^{\xi_- t} \Rightarrow A(\xi_+ - \xi_-) = 1$ for $\hat{u}_t(\eta,0) = 1$

Get $A = \dfrac{1}{\sqrt{1-4\eta^2}}$ & $\boxed{\hat{u}(\eta,t) = \dfrac{1}{\sqrt{1-4\eta^2}}\left(e^{\xi_+ t} - e^{\xi_- t}\right)}$

Q: How to understand the sol. $\hat{u}(\eta,t)$?

A: Need to find sol. in x-space.

Intuition: $f = \dfrac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\eta)\cdot e^{i\eta x}\, d\eta$ wave number

dispersion relationship.

low pitch: η small, $1 - 4\eta^2 > 0$ (smooth). by Talyor's expansion of $\dfrac{-1+\sqrt{1-4\eta^2}}{2} \approx \dfrac{-1 + (1-4\eta^2)^{1/2}}{2} + O(1)\eta^4$

$$e^{\xi_+ t} = e^{\frac{-1+\sqrt{1-4\eta^2}}{2} t}$$

$$\Rightarrow e^{\xi_+ t} \sim e^{-2\eta^2 t} \Rightarrow \sim \frac{1}{\sqrt{8\pi t}} \cdot e^{-\frac{x^2}{8t}}$$

Task 1: Extract Singularity in the frequence domain.

① Compute

$$\begin{cases}\hat{u}(\eta,t)=\dfrac{1}{\sqrt{1-4\eta^2}}\left(e^{\xi_+ t}-e^{\xi_- t}\right) \\ \xi_\pm=\dfrac{-1\pm\sqrt{1-4\eta^2}}{2}\end{cases};$$

Try to understand the sol. as $\eta\to\infty$;

$$\sqrt{1-4\eta^2}=2\eta i\sqrt{1-\frac{1}{4\eta^2}}\ \xrightarrow[\text{expansion}]{\text{Talyor's}}\ i2\eta\left(1-\frac{1}{8\eta^2}+\frac{O(1)}{\eta^4}\right)$$

$$= i2\eta\left(1-\frac{1}{8\eta^2}+\frac{C}{\eta^4}+\frac{O(1)}{\eta^6}\right)$$

Now, find $\xi_\pm^*$ to approximate $\xi_\pm$:

② Define

$$\boxed{\xi_\pm^*=\frac{-1\pm 2\eta i\left(1-\frac{1}{8(\eta^2+1)}+\frac{C}{(\eta^2+1)^2}\right)}{2}}$$

& $|\xi_\pm-\xi_\pm^*|=O(1)\cdot\frac{1}{\eta^5}$ as $\eta\to\infty$

Want to find approximating sol. of $\hat{u}(\eta,t)$:

Notice $(\sqrt{1-4\eta^2})^*=2\eta i\left(1-\frac{1}{8(\eta^2+1)}+\frac{C}{(\eta^2+1)^2}\right)$

$$\Downarrow\quad \boxed{\hat{u}^*(\eta,t)=\frac{1}{(\sqrt{1-4\eta^2})^*}\cdot\left(e^{\xi_+^* t}-e^{\xi_-^* t}\right)}$$

③ Define $\hat{u}^*(\eta,t)$

The initial conditions are satisfied as follows:

- $\hat{u}_t^*(\eta,0)=1$, $\hat{u}^*(\eta,0)=0$;
- $\hat{u}_{tt}^*+\eta^2\hat{u}^*+\hat{u}_t^*$

$$=\left[(\xi_+^*)^2\cdot e^{\xi_+^* t}-(\xi_-^*)^2\cdot e^{\xi_-^* t}+\eta^2 e^{\xi_+^* t}-\eta^2 e^{\xi_-^* t}+\xi_+^*\cdot e^{\xi_+^* t}-\xi_-^* e^{\xi_-^* t}\right]\frac{1}{(\sqrt{1-4\eta^2})^*}$$

$$=\frac{1}{(\sqrt{1-4\eta^2})^*}\left[\left((\xi_+^*)^2+\eta^2+\xi_+^*\right)\cdot e^{\xi_+^* t}-\left((\xi_-^*)^2+\eta^2+\xi_-^*\right)\cdot e^{\xi_-^* t}\right]$$

Note: $[(\xi_+^*)^2 + \eta^2 + \xi_+^*] \cdot e^{\xi_+^* t}$

$$= (\cancelto{0}{\xi_+^2 + \eta^2 + \xi_+} + \underbrace{((\xi_+^*)^2 - \xi_+^2) + (\xi_+^* - \xi_+)}_{O(1)\frac{1}{\eta^5}}) \cdot e^{\xi_+^* \cdot t}$$

for $|\xi_\pm - \xi_\pm^*| = O(1)\frac{1}{\eta^5}$

$$\Rightarrow \hat{u}^*_{tt} + \eta^2 \hat{u}^* + \hat{u}^*_t = \hat{S}(\eta, t) \le O(1) \cdot \frac{e^{-t}}{(|\eta|+1)^5}$$

Compare $|\hat{u}^*(\eta,t) - \hat{u}(\eta,t)|$:

$$\Rightarrow \begin{cases} (\hat{u} - \hat{u}^*)_{tt} + \eta^2(\hat{u} - \hat{u}^*) + (\hat{u} - \hat{u}^*)_t = O(1) \cdot \dfrac{e^{-t}}{c(|\eta|+1)^5} \\ (\hat{u} - \hat{u}^*)(\eta, 0) = (\hat{u}_t - \hat{u}_t^*)(\eta, 0) = 0 \end{cases}$$

Let $\hat{V} = \hat{u} - \hat{u}^*$

$$(*) \begin{cases} \hat{V}_{tt} + \eta^2 \hat{V} + \hat{V}_t = O(1) \cdot \dfrac{e^{-t}}{(|\eta|+1)^5} \le e^{-t} \\ \hat{V}(\eta, 0) = \hat{V}_t(\eta, 0) = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} |\partial_x^3 S(x,t)|^2 dx \le e^{-t}$$

Rewrite the $\#$ as:

$$\begin{cases} v_{tt} - v_{xx} + v_t = S(x,t) \\ v(x,0) = v_t(x,0) = 0 ; \end{cases}$$

Energy Estimate:

$$\int_{\mathbb{R}} v_t \cdot (v_{tt} - v_{xx} + v_t)\, dx = \int_{\mathbb{R}} S(x,t)\, v_t\, dx$$

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(v_t)^2 dx + \int_{\mathbb{R}} \underbrace{v_{tx} v_x}_{\left(\frac{v_x^2}{2}\right)_t} dx + \int_{\mathbb{R}} v_t^2 dx = \int_{\mathbb{R}} S(x,t)\, v_t\, dx$$

$$\Rightarrow \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(v_t^2 + v_x^2)\, dx + \int_{\mathbb{R}} v_t^2 dx = \int_{\mathbb{R}} S(x,t)\, v_t\, dx$$

Integrate on t:

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{2}[(v_t)^2 + (v_x)^2]\, dx \Big|_0^T + \int_0^T \int_{\mathbb{R}} (v_t)^2 dx\, dt = \int_0^T \int_{\mathbb{R}} S(x,t)\, v_t(x,t)\, dx\, dt$$

$$\le \int_0^T \int_{\mathbb{R}} \frac{S^2(x,t) + (v_t)^2}{2}\, dx\, dt$$

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}} (v_t)^2 dx\, dt$$

$$(v_t)^2 \ge 0$$

$$\| \quad \int_0^T \int_{\mathbb{R}} \frac{S^2(x,t)}{2}\, dx\, dt$$

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{2}[v_t^2 + v_x^2]\, dx \Big|_0^T + \frac{1}{2}\int_0^T \int_{\mathbb{R}} v_t^2\, dx\, dt$$

$$\le \int_0^T \int_{\mathbb{R}} \frac{S^2(x,t)}{2}\, dx\, dt$$

Similarly,

$$\int_{\mathbb{R}} \partial_x^k v_t \cdot \partial_x^k (v_{tt} - v_{xx} + v_t)\, dx = \int_{\mathbb{R}} \partial_x^k S(x,t) \cdot \partial_x^k v_t\, dx$$

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{2} [(\partial_x^k v_t)^2 + (\partial_x^k v_x)^2 \, dx \Big|_0^T + \frac{1}{2} \int_0^T \int_{\mathbb{R}} (\partial_x^k v_t)^2 \, dx\, dt$$

$$\leq \int_0^T \int_{\mathbb{R}} \frac{(\partial_x^k S(x,t))^2}{2} dx\, dt$$

$k = 1, 2, 3.$

Task 2: Find the structure $u^*(x,t)$.

Another Functional Thm:

If $f(x)$ satisfies $\begin{cases} \hat{f}(\eta): \text{Analytic in } |\operatorname{Im}\eta| < \eta_0 \\ |\hat{f}(\eta)| \lesssim \dfrac{O(1)}{\eta^2+1} \end{cases}$

Then, $|f(x)| \lesssim O(1)\cdot e^{-\eta_0 |x|}$

Proof:

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta x}\,\hat{f}(\eta)\,d\eta \underset{\eta=\xi+iv_0}{\overset{\text{Cauchy Integrable Thm.}}{=\!=}} \frac{1}{2\pi}\int_{\mathbb{R}+iv_0} e^{i\eta x}\hat{f}(\eta)\,d\eta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi+iv_0)x}\hat{f}(\xi+iv_0)\,d\xi = \frac{e^{-v_0 x}}{2\pi}\int_{-\infty}^{\infty} e^{i\xi}\cdot\hat{f}(\xi+iv_0)\,d\xi$$

$|v_0| < \eta_0$

$$\Rightarrow |f(x)| \le \frac{e^{-v_0 x}}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi+iv_0)|\,d\xi$$

$$\lesssim \frac{e^{-v_0 x}}{2\pi}\, C \quad \leftarrow \text{const.}$$

$$\Rightarrow |f(x)| \le \frac{e^{-|v_0 x|}}{2\pi}\cdot C \lesssim O(1)\cdot e^{-\eta_0|x|}$$

Q.E.D.

Recall,

$$\begin{cases}
\bullet\ \xi_\pm^* = \dfrac{-1 \pm 2i\eta\left[1 - \frac{1}{8(\eta^2+1)} + \frac{c}{(\eta^2+1)^2}\right]}{2} \\
\bullet\ \left(\sqrt{1-4\eta^2}\right)^* = i2\eta\left(1 - \frac{1}{8(\eta^2+1)} + \frac{c}{(\eta^2+1)^2}\right) \\
\bullet\ |\xi_\pm - \xi_\pm^*| = O(1)\dfrac{1}{\eta^6}, \quad \text{as } \eta \to \infty \\
\bullet\ \hat{u}^*(\eta,t) = \dfrac{1}{\left(\sqrt{1-4\eta^2}\right)^*}\cdot\left(e^{\xi_+^* t} - e^{\xi_-^* t}\right)
\end{cases}$$

analytic in $|\operatorname{Im}(\eta)| < \alpha/2$

$$\mathrm{I} \longrightarrow \quad = \frac{e^{-\frac{t}{2}}\left[e^{2i\eta\left[1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right]t} - e^{-2i\eta\left[1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right]} \right]}{2i\eta\cdot\left(1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right)}$$

$$\mathrm{II} \longrightarrow \quad 2i\eta\cdot\left(1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right)$$

Consider

$$e^{2i\eta\cdot\left[1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right]t}$$

$$= \underbrace{e^{2i\eta t}}_{\delta(x+2t)}\cdot \underbrace{e^{-\frac{2i\eta t}{8(\eta^2+1)}+\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}}}$$

Reconsider:

$$\underbrace{e^{-\frac{2i\eta t}{8(\eta^2+1)}+\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}} - 1}_{\delta(x)} = \int_0^{-\frac{2i\eta t}{8(\eta^2+1)}+\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}} e^{\alpha}\, d\alpha$$

$$= \underbrace{-\frac{2i\eta t}{8(\eta^2+1)}} + \underbrace{\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}}_{e^{-|\eta_0 x|}} + O(1)\,t^2$$

By Complex Analysis:

$$\frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}\frac{1}{1+\eta^2}\cdot e^{i\eta x}\,d\eta$$

$$= \frac{1}{2\pi}\cdot \mathrm{Res.}\left(\frac{e^{i\eta x}}{1+\eta^2}\right)$$

$$= \frac{1}{2\pi}\cdot 2\pi i\cdot\frac{e^{-x}}{2i} = k\cdot e^{-x}, \text{ if } x>0$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{1+\eta^2}e^{i\eta x}\,dx = c\cdot e^{-|x|}.$$

Thus, $\quad \dfrac{-2i\eta}{8(\eta^2+1)} \longleftarrow c\cdot t\cdot\left(e^{-|x|}\right)_x.$

Then,

$$\mathrm{I} = e^{-t/2}\cdot\Big[\,\delta(x+2t) * \left[\left(e^{-|x|}\right)_x + e^{-|\eta_0 x|}\right]c\cdot t.$$
$$- \delta(x-2t) * \left[\left(e^{-|x|}\right)_x + e^{-|\eta_0 x|}\right]c\cdot t\,\Big]$$

II is a polynomial $\Rightarrow u^*(x,t)$ is around $e^{-|x|-t/k}.$

Task 3: Verify $|u(x,t) - u^*(x,t)|$ is exponentially small in the region $|x| > \alpha t$ for any positive α.

By Energy Estimate on $v(x,t) = u(x,t) - u^*(x,t)$.

- $\int_{-\infty}^{\infty} (v_t^2(x,t) + v_x^2(x,t)) \cdot e^{\varepsilon(x - \alpha t)} dx \le O(1) \cdot e^{-\varepsilon^2 t}$ for $x > \alpha t$
- $\int_{-\infty}^{\infty} (v_t^2(x,t) + v_x^2(x,t)) \cdot e^{-\varepsilon(x + \alpha t)} dx \le O(1) \cdot e^{-\varepsilon^2 t}$ for $x < -\alpha t$

Recall, $\begin{cases} v_{tt} - v_{xx} + v_t = s(x,t), \\ v(x,0) = v_t(x,0) = 0 \end{cases}$

$|\partial_x^3 s(x,t)|^2 < O(1) \cdot e^{-t - |x|/k}$ for $|\hat{s}(\eta,t)| \le \frac{e^{-t}}{(|\eta|+1)^4} \cdot O(1)$

Observe:

If $|x| > \frac{\alpha t}{2}$, then $\exists$ k_1 s.t.

$$|v(x,t)| \le O(1) \cdot e^{-(t+|x|)/k_1}$$

By method of energy:

$\forall \varepsilon > 0$,

$$\int_{-\infty}^{\infty} e^{\varepsilon(x - \frac{\alpha}{2}t)} \cdot v_t \cdot (v_{tt} - v_{xx} + v_t)\, dx = \int_{-\infty}^{\infty} s(x,t) \cdot e^{\varepsilon(x - \frac{\alpha}{2}t)} \cdot v_t\, dx$$

- W.T. know how $U = u^* - u$ looks like?

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0 \\ u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{cases} \quad \& \quad \begin{cases} u^*_{tt} - u^*_{xx} + u^*_t = S(x,t). \\ u^*(x,0) = 0 \\ u^*_t(x,0) = \delta(x). \end{cases}$$

Combine the two equations:

$$\Rightarrow \begin{cases} V_{tt} - V_{xx} + V_t = -S(x,t) \\ V(x,0) = 0 \\ V_t(x,0) = 0, \end{cases} \qquad |\partial_x^3 S(x,t)| \le O(1) \cdot e^{-t - \frac{|x|}{K}}$$

If $|x| > t/2$, there exists k_1 s.t. $|V(x,t)| \le k_1 \cdot e^{-(|x|+t)/k_1}$

What we want to prove

Proof: Consider $e^{\varepsilon(x - \frac{1}{8}t)}$,

$$\int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t \cdot (V_{tt} - V_{xx} + V_t)\, dx = \int_{-\infty}^{\infty} -S(x,t) \cdot e^{\varepsilon(x-\frac{t}{8})} \cdot V_t\, dx$$

(I) (II) (III)

$$V_t \cdot V_{tt} = \left(\frac{1}{2} V_t^2\right)_t$$

$$\Downarrow \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot \left(\frac{V_t^2}{2}\right)_t dx = \frac{d}{dt}\Big[\frac{1}{2}\int e^{\varepsilon(x-\frac{1}{8}t)} \cdot (V_t)^2\, dx + \frac{\varepsilon}{16}\int e^{\varepsilon(x-\frac{t}{8})} \cdot V_t^2\, dx$$

$$\bullet\ -\int e^{\varepsilon(x-\frac{t}{8})} \cdot V_t \cdot V_{xx}\, dx \overset{\text{I.B.P}}{=} \int_{-\infty}^{\infty} \left(e^{\varepsilon(x-\frac{t}{8})} \cdot V_t\right)_x \cdot V_x\, dx$$

$$= \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{t}{8})} \cdot V_{tx} V_x\, dx + \varepsilon\int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{t}{8})} \cdot V_t \cdot V_x\, dx$$

$$= \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot \left(\frac{V_x^2}{2}\right)_t dx + \varepsilon \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t \cdot V_x \, dx$$

$$= \frac{d}{dt} \frac{1}{2} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2 \, dx + \frac{\varepsilon}{16} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2 \, dx + \bigcirc$$

& (III) $\Rightarrow \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t^2 \, dx$.

In all,

$$\boxed{\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int e^{\varepsilon(x-\frac{1}{8}t)} \cdot (V_t)^2 dx + \frac{\varepsilon}{16} \int e^{\varepsilon(x-\frac{1}{8}t)} \cdot (V_t)^2 dx \\ &+ \frac{1}{2}\frac{d}{dt} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2 + \frac{\varepsilon}{16} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2 dx + \varepsilon \int_{-\infty} e \cdot V_t V_x dx \\ &+ \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t^2 dx = \int_{-\infty}^{\infty} -S(x,t) \, e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t \, dx \end{aligned}}$$

If we choose ε small enough, then

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{1}{16}\varepsilon V_x^2 + \varepsilon V_t V_x + V_t^2\right) \cdot e^{\varepsilon(x-\frac{1}{8}t)} dx$$

$$\geq \frac{1}{32}\varepsilon \cdot (V_x^2 + V_t^2)$$

Rewrite:

$$\Rightarrow \frac{1}{2}\frac{d}{dt} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot (V_t^2 + V_x^2) dx + \int_{-\infty}^{\infty} \frac{1}{32}\varepsilon (V_x^2 + V_t^2) \cdot e^{\varepsilon(x-\frac{1}{8}t)}$$

$$\leq \left| - \int_{-\infty}^{\infty} S(x,t) \cdot e^{\varepsilon(x-\frac{1}{8}t)} V_t \, dx \right|$$

$$\leq \int_{-\infty}^{\infty} \left[\frac{S^2(x,t)}{1/64 \, \varepsilon} + \frac{1}{64}\varepsilon (V_t)^2 \right] e^{\varepsilon(x-\frac{1}{8}t)} dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt} \int_{-\infty}^{\infty} \cdot e^{\varepsilon(x-\frac{1}{8}t)} \cdot [(V_t)^2 + (V_x)^2] dx + \int_{-\infty}^{\infty} \frac{1}{64}\varepsilon \cdot (V_x^2 + V_t^2) \cdot e^{\varepsilon(x-\frac{1}{8}t)} dx$$

$$\leq \int_{-\infty}^{\infty} \frac{S^2(x,t)}{1/64 \cdot \varepsilon} \cdot e^{\varepsilon(x-\frac{1}{8}t)} dx.$$

$$\Rightarrow \int_{\mathbb{R}} e^{\varepsilon(x-\frac{1}{8}t)} [\cdot V_t^2 + V_x^2] dx \Big|_{t=T} \leq \int_0^T \int_{-\infty}^{\infty} O(1) \cdot S^2 \cdot e^{\varepsilon(x-\frac{1}{8}t)} dx \cdot dt$$

Recall, $\hat{S}(\eta, t) = O(1) \cdot \frac{e^{-t}}{(|\eta|+1)^5}$

i.e. $S^2(x,t) = O(1) \cdot e^{-(t+|x|)/k}$.

$$\Rightarrow \int_{\mathbb{R}} e^{\varepsilon(x-\frac{1}{8}t)} \cdot [V_t^2 + V_x^2] dx \Big|_{t=T} \leq O(1) \cdot e^{-\frac{\varepsilon T}{8}}$$

Class II:

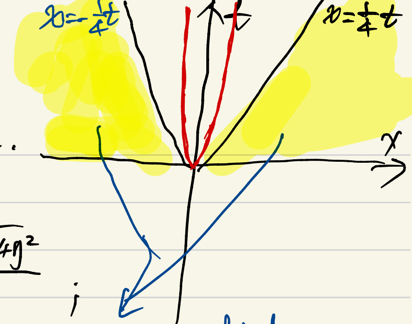
Task 4: By Long wave – Short wave Decomposition:

$$u_L(x,t) = \int_{|\eta|<\varepsilon} e^{i\eta x} \cdot \hat{u}(\eta, t)\, d\eta$$

$$u_S(x,t) = \int_{|\eta|>\varepsilon} \cdot e^{i\eta x} \cdot \hat{u}(\eta, t)\, d\eta;$$

$\mathrm{Re}(\beta_\pm(\eta)) < -\varepsilon_0 \quad \Rightarrow \quad \|u_S\|_{L^2} \leq O(1) \cdot e^{-\varepsilon_0 t}$

$\Rightarrow \|u_S\|_\infty \leq O(1) \cdot e^{-\varepsilon_0 t}$.



Aim: Want to know what happens in the cone.

Recall,

$$\hat{u}(\eta, t) = \frac{1}{\sqrt{1-4\eta^2}} \left(e^{\xi_+ t} - e^{\xi_- t}\right), \quad \xi_\pm = \frac{-1 \pm \sqrt{1-4\eta^2}}{2};$$

$\eta :\sim$ wave # (ossilation)

exponential decays w.r.t. (t, x)

Long wave - short wave decomposition:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta x} \hat{u}(\eta, t)\, d\eta,$$

- $|\eta| >> 1$ — short wave (wave length small)
- $|\eta| << 1$ — Long wave (wave length big)

$$u(x,t) = \frac{1}{2\pi}\Big(\underbrace{\int_{|\eta|<\varepsilon}}_{\text{"Long wave component"}} + \underbrace{\int_{|\eta|>\varepsilon}}_{\text{"short wave component"}}\Big) \cdot e^{i\eta x} \cdot \hat{u}(\eta, t)\, d\eta$$

$$= u_L(x,t) + u_S(x,t).$$

For $|\eta| > \varepsilon$, $\mathrm{Re}(\xi_-) < -\frac{1}{2} \Rightarrow |e^{\xi_-(\eta)t}| < e^{-\frac{1}{2}t}$

$$\mathrm{Re}(-1 + \sqrt{1-4\eta^2}) = \mathrm{Re}\left(\frac{1-(1-4\eta^2)}{-1-\sqrt{1-4\eta^2}}\right)$$

$$= \mathrm{Re}\left(-\frac{4\eta^2}{1+\sqrt{1-4\eta^2}}\right) \leq -\frac{4\varepsilon^2}{2}$$

$$\Rightarrow |e^{-\xi_\mp(\eta)t}| \leq O(1)\, e^{-2\varepsilon^2 t}$$

$$|u_S(x,t)| = \left|\frac{1}{2\pi} \int_{|\eta|>\varepsilon} e^{i\eta x} \cdot \frac{1}{\sqrt{1-4\eta^2}} \left(\cdot e^{\xi_+ t} - e^{\xi_- t}\right) d\eta\right|$$

Recall

$\|u_S(\cdot,t)\|_{L^2}^2$: $f \in L^2(\Omega) \Rightarrow \|\hat{f}\|_{L^2} = \|f\|_{L^2}$

$$\int |\hat{f}|^2 = \int |f|^2$$

Thus, $$\|u_S(\cdot,t)\|_{L^2}^2 = \int_{|\eta|>\varepsilon} \frac{1}{|1-4\eta^2|} \cdot |e^{\xi_+ t} - e^{\xi_- t}|^2 \, d\eta$$
$$\leq O(1) \cdot e^{-\varepsilon^2 t/2}$$

$\|u_S^* - u_S\|_\infty < O(1) \cdot e^{-\varepsilon^2 t/2}$ for all x.

To find $u_L(x,t)$ with $x < \frac{1}{2}t$

$$u_L(x,t) = \frac{1}{2\pi} \int_{|\eta|<\varepsilon} e^{i\eta x} \cdot \frac{1}{\sqrt{1-4\eta^2}} \cdot (e^{\xi_+(\eta)t} - e^{\xi_-(\eta)t}) \, d\eta$$

Analytic & Cachy formula

$$= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{e^{i\eta x}}{\sqrt{1-4\eta^2}} (e^{\xi_+ t} - e^{\xi_- t}) \, d\eta$$

$$\xi_\pm(\eta) = \frac{-1 \pm \sqrt{1-4\eta^2}}{2} = \frac{-1 \pm ((1-2\eta^2) + Q(\eta^4))}{2};$$ Q : Analytic around 0

$$\begin{cases} \xi_- = -1 + \eta^2 - Q(\eta^4) \Rightarrow \operatorname{Re}(\xi_-(\eta)) \leq -\frac{1}{2} \text{ for } |\eta| < \varepsilon \\ \xi_+ = -\eta^2 + O(\eta^4) \text{ — Problem for } |\eta| \text{ small enough} \end{cases}$$

Focus on ξ_+:

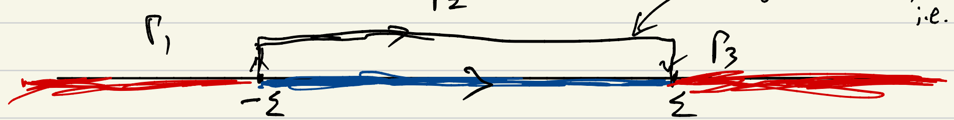
$$\frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{e^{i\eta x + \xi_+(\eta) t}}{\sqrt{1-4\eta^2}} \, d\eta$$

$$= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{e^{i\eta x + (-\eta^2 + O(\eta^4))t + \frac{x^2}{4t} - \frac{x^2}{4t}}}{\sqrt{1-4\eta^2}} \, d\eta$$

$$= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{1-4\eta^2}} \cdot e^{-\frac{x^2}{4t} - t(\eta - \frac{ix}{2t})^2 + O(\eta^4) t} \, d\eta$$

How to balance the term?

$\mathrm{Im}(\eta) = \gamma\frac{x}{2t}$, γ small. i.e. $0<\gamma\ll 1$.



$$\frac{1}{2\pi}\int_{\Gamma_2} e^{-\frac{x^2}{4t} - t(\eta-\frac{ix}{2t})^2 + O(\eta^4)t}\,d\eta, \quad \eta = \nu + i\cdot\gamma\cdot\frac{x}{2t}, \quad \gamma\in(-\varepsilon,\varepsilon)$$

$$= \frac{1}{2\pi}\int_{\Gamma_2} e^{-\frac{x^2}{4t} - t(\nu + i\frac{\gamma x}{2t} - \frac{ix}{2t})^2 + O((\nu+i\gamma\frac{x}{2t})^4)t}\,d\nu$$

$$= \frac{1}{2\pi}\int_{\Gamma_2} e^{-\frac{x^2}{4t} - t(\nu + (1-\gamma)i\frac{x}{2t})^2 + O((\nu+i\gamma\frac{x}{2t})^4)t}\,d\nu$$

$$-\frac{x^2}{4t} - t[\nu - i(1-\gamma)\frac{x}{2t}]^2 = -(1-(1-\gamma)^2)\frac{x^2}{4t} - t\nu^2 - \{t\nu(1-\gamma)\frac{x}{2t}\}.$$

$$= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{-(1-(1-\gamma)^2)\frac{x^2}{4t}}\cdot e^{-t\nu^2 - \gamma t\nu(1-\gamma)\frac{x}{2t} + tO((\nu+\frac{ix}{2t}\gamma)^4)}\,d\nu$$

$$-\frac{t\nu^2}{2} - \frac{t\nu^2}{2}$$

$$= \frac{1}{2\pi}\cdot e^{-(1-(1-\gamma)^2)\frac{x^2}{4t}}\cdot\int_{-\varepsilon}^{\varepsilon} e^{\underbrace{-t\nu^2 - \gamma t\nu(1-\gamma)\frac{x}{2t} + tO((\nu+\frac{ix}{2t}\gamma)^4)}}\,d\nu$$

$$= \frac{e^{-[1-(1-\gamma)^2]\frac{x^2}{8t}}}{2\pi}\cdot\int_{-\varepsilon}^{\varepsilon} \left(e^{-[1-(1-\gamma)^2]\frac{x^2}{8t}} \cdots \right) d\nu$$

$$\mathrm{Re}\left(-[1-(1-\gamma)^2]\frac{x^2}{8t} - \frac{t\nu^2}{2} + O((\nu+\frac{ix}{2t}\gamma)^4)t\right) < 0$$

if $\nu\in(-\varepsilon,\varepsilon]$, $\gamma \ll 1$

$$\leq \frac{e^{-[1-(1-\gamma)^2]\frac{x^2}{8t}}}{2\pi}\cdot\int_{-\varepsilon}^{\varepsilon} e^{-\frac{t\nu^2}{2}}\,d\nu, \qquad d\nu = \frac{d\,\nu\sqrt{t}}{\sqrt{t}}.$$

$$\leq O(1)\cdot\frac{e^{-[-(1-\gamma)^2]\frac{x^2}{8t}}}{\sqrt{t}}$$

1. $\begin{cases} V_{tt} - V_{xx} + V_t = 0 & , \ x \in R \\ V(x,0) = 0 \\ V_t(x,0) = \delta(x - x_0) & , \ x_0 > 0 \end{cases}$

Take Laplace Transform:

$$s^2 \mathcal{L} u - \delta(x - x_0) - \partial_x^2 \mathcal{L} u + s \mathcal{L} u = 0. \quad (x \in R)$$

$$\Rightarrow \quad (s^2 + s) \cdot \mathcal{L} u - \partial_x^2 \mathcal{L} u = \delta(x - x_0)$$

Then, $\mathcal{L} u = \begin{cases} A \cdot e^{-\sqrt{s^2+s}\,(x - x_0)} & \text{if } x > x_0 \\ A \cdot e^{\sqrt{s^2+s}\,(x - x_0)} & \text{if } x < x_0 \end{cases}$

$$\mathcal{L}_x u = \begin{cases} -\sqrt{s^2+s} \cdot A \cdot e^{-\sqrt{s^2+s}\,(x - x_0)} & , \text{ if } x > x_0 \\ \sqrt{s^2+s} \cdot A \cdot e^{\sqrt{s^2+s}\,(x - x_0)} & , \text{ if } x < x_0 \end{cases}$$

$$\Rightarrow \quad A = \frac{1}{2\sqrt{s^2+s}}$$

$$\Rightarrow \quad \mathcal{L} u = \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}\,|x - x_0|}$$

2. $\begin{cases} V_{tt} - U_{xx} + V_t = 0, & x > 0 \\ V(x,0) = 0, \ V_t(x,0) = \delta(x - x_0), & x_0 > 0 \\ V(0,t) = 0 \end{cases}$

$$\Rightarrow \begin{cases} s^2 \mathcal{L} V - \delta(x - x_0) - \partial_x^2 \mathcal{L} V + s \mathcal{L} V = 0 & (x > 0) \\ \mathcal{L} V(0,s) = 0 \end{cases}$$

$$\Rightarrow \quad s^2 \mathcal{L}u - \delta(x-x_0) - \partial_x^2 \mathcal{L}u + s\mathcal{L}u = 0$$

Then, $\quad s^2\mathcal{L}(v-u) - \partial_x^2 \mathcal{L}(v-u) + s\mathcal{L}(v-u) = 0$

$$\left\{ \mathcal{L}(v-u)(0,s) = -\mathcal{L}u(0,s) \right.$$

$$\Rightarrow \mathcal{L}(v-u)(x,s) = -\mathcal{L}u(0,s) \cdot e^{-\sqrt{s^2+s}\,x}$$

$$\mathcal{L}u = \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}\,|x-x_0|}$$

$$\Rightarrow \mathcal{L}v = \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}\,|x-x_0|} - \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}\,(x_0+x)}$$

3. $$\begin{cases} v_{tt} - v_{xx} + v_t = 0, & x > 0 \\ v(x,0) = 0, \quad v_t(x,0) = \delta(x-x_0), & x_0 > 0 \\ v(0,t) + v_x(0,t) = 0 \end{cases} \Rightarrow (\mathcal{L}v + \partial_x \mathcal{L}v)(0,s) = 0$$

$$\Rightarrow \begin{cases} s^2\mathcal{L}v - \delta(x-x_0) - \partial_x^2 \mathcal{L}v + s\mathcal{L}v = 0 & (x > 0) \\ (\mathcal{L}v + \partial_x \mathcal{L}v)(0,s) = 0 \end{cases}$$

$$\Rightarrow \quad s^2\mathcal{L}u - \delta(x-x_0) - \partial_x^2 \mathcal{L}u + s\mathcal{L}u = 0$$

$$\mathcal{L}(v-u)(x,s) = Q(s) \cdot e^{-\sqrt{s^2+s}\,x}$$

$$(\mathcal{L} + \partial_x\mathcal{L})(v-u)(0,s) = -(1+\partial_x)\mathcal{L}u(0,s)$$

$$\Rightarrow (1 - \sqrt{s^2+s})\,Q = (1 + \sqrt{s^2+s}) \cdot \frac{e^{-\sqrt{s^2+s}\cdot x_0}}{2\sqrt{s+s^2}}$$

$$\Rightarrow \quad Q = -\left(\frac{1+\sqrt{s^2+s}}{1-\sqrt{s^2+s}}\right) \cdot e^{-\sqrt{s^2+s}\cdot x_0}$$

$$\Rightarrow \mathcal{L} V = \mathcal{L} u + \frac{1}{s^2+s-1} \cdot \underbrace{(1+\sqrt{s^2+s})^2}_{(1-\partial x)^2} \cdot \frac{e^{-\sqrt{s^2+s}\,(x+x_0)}}{2\cdot\sqrt{s+s^2}}$$

$$V = u + \mathcal{L}^{-1}\left(\frac{1}{s^2+s-1}\right) * (1-\partial x)^2\, u(x-2x_0, t).$$