High dimension problem:

$$u_{tt} - \Delta u = 0, \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad (x, y, z) \in \mathbb{R}^3$$

$$\vec{\eta} = (\eta_1, \eta_2, \eta_3)$$

$$\hat{u}(\vec{\eta}, t) = \iiint_{\mathbb{R}^3} e^{-\vec{\eta}\cdot\vec{x}\, i}\, u(\vec{x}, t)\, d\vec{x} \longleftarrow \text{Fourier transform.}$$

then,
$$\hat{u}_{tt} - \iiint_{\mathbb{R}^3} e^{-i\vec{\eta}\cdot\vec{x}} \cdot (\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot u(\vec{x}, t)\, d\vec{x} = 0$$

$$\xRightarrow{\text{I.B.P}}$$

$$\hat{u}_{tt} - \iiint_{\mathbb{R}^3} [(\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot e^{-i\vec{\eta}\cdot\vec{x}}] \cdot u(\vec{x}, t)\, d\vec{x} = 0$$

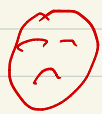
$$\hat{u}_{tt} + |\vec{\eta}|^2 \hat{u} = 0 \quad \longleftarrow \text{By computation.}$$

$$\Rightarrow \hat{u}(\vec{\eta}, t) = \frac{1}{2}\left(e^{i|\vec{\eta}|t} + e^{-i|\vec{\eta}|t}\right)\hat{u}(\eta, 0) + \frac{e^{i|\vec{\eta}|t} - e^{-i|\vec{\eta}|t}}{2i|\vec{\eta}|} \cdot \hat{u}_t(\vec{\eta}, 0).$$

Inverse Fourier Transform,

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\left(e^{i|\vec{\eta}|t} + e^{-i|\vec{\eta}|t}\right)}{2} \cdot e^{i\vec{\eta}\cdot\vec{x}}\, d\vec{\eta} \longleftarrow \text{Q: ? Can't be handled.}$$

Fourier Transform Method fails !!!



Possion mean:

For 1-D:

$$w_{tt} - w_{xx} = 0, \quad x \in \mathbb{R}.$$

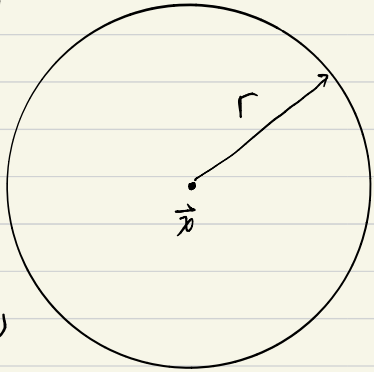
$$\Rightarrow w(x, t) = \frac{1}{2}[w(x+t, 0) + w(x-t, 0)] + \frac{1}{2}\int_{x-t}^{x+t} w_t(x, 0)\, dx.$$

by Taylor' expansion.

$$\begin{cases} u_{tt} - \Delta u = 0, & \Delta = \nabla \cdot \nabla \\ u(\vec{x}, t): \end{cases}$$

Define $\bar{u}(\vec{x}, t; r) = \dfrac{1}{4\pi r^2} \cdot \displaystyle\int_{\partial B_r(\vec{x})} u(\vec{y}, t)\, dA_y$

Now, $\displaystyle\int_{B_r(\vec{x})} u_{tt}(\vec{y}, t)\, d\vec{y} = \int_{B_r(\vec{x})} \nabla \cdot \nabla u(\vec{y}, t)\, d\vec{y}$



Divergence thm

$$= \int_{\partial B_r(\vec{x})} \nabla u(\vec{y}, t) \cdot \vec{n}\, ds$$

$$= \int_{\partial B_r(\vec{x})} \frac{\partial u}{\partial r}\, ds, \qquad dA = r^2 \cdot \underbrace{dS_1}_{\partial B_1(\vec{x})}$$

$$= \int_{B_1(\vec{x})} \frac{\partial u}{\partial r} \cdot r^2\, dS_1$$

$$= r^2 \cdot 4\pi \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r)$$

$$\iiint_{B_r(\vec{x})} u_{tt}(\vec{y}, t)\, d\vec{y} = \int_0^r \iint_{|w|=1} u_{tt}(\rho\vec{w} + \vec{x}, t) \cdot \rho^2\, d\Omega\, d\rho$$

$$= \int_0^r 4\pi \rho^2\, \bar{u}_{tt}(\vec{x}, t; \rho)\, d\rho$$

Now, we have:

$$r^2 \cdot 4\pi \cdot \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r) = \int_0^r 4\pi\rho^2\, \bar{u}_{tt}(\vec{x}, t; \rho)\, d\rho$$

Take derivative w.r.t. r,

$$\frac{\partial}{\partial r}\left(r^2 \cdot 4\pi \cdot \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r)\right)$$

$$= 4\pi r^2\, \bar{u}_{tt}(\vec{x}, t; r)$$

Simplified:

$$2r \cdot \frac{\partial \bar{u}}{\partial r} + r^2 \cdot \frac{\partial^2 \bar{u}}{\partial r^2} = r^2 \, \bar{u}_{tt}$$

$$\Rightarrow \bar{u}_{tt} = \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \bar{u}}{\partial r} \quad \leftarrow (*)$$

Then, $(*) \cdot r$

$$\Rightarrow \boxed{(r \cdot \bar{u})_{tt} = (r\bar{u})_{rr}}$$ with $r > 0$.

Next, by odd extension for $r \cdot \bar{u} = 0$ at $r = 0$

$$\Rightarrow (r\bar{u}) = \frac{(r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t)\,\bar{u}(\bar{x}, 0; r-t)}{2} + \frac{1}{2}\int_{r-t}^{r+t} \rho \, \bar{u}_t(x, 0; \rho)\, d\rho$$

Simplified:

$$\bar{u}(\bar{x}, t; r) = \frac{(r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t)\,\bar{u}(\bar{x}, 0; r-t)}{2r} + \frac{1}{2r}\int_{r-t}^{r+t} \rho \, \bar{u}_t(x, 0; \rho)\, d\rho$$

Recall, for contin. solution,

$$\lim_{r \to 0} \bar{u}(\bar{x}, t; r) = u(\bar{x}, t)$$ by Lebegues Thm.

Then, $\lim_{r \to 0} \bar{u}(\vec{x}, t; r)$

$$= \lim_{r \to 0} \frac{(r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t)\,\bar{u}(\bar{x}, 0; r-t)}{2 \cdot r} \leftarrow (\mathrm{II})$$

$$+ \underbrace{\lim_{r \to 0} \frac{1}{2r}\int_{r-t}^{r+t} \rho \, \bar{u}_t(x, 0; \rho)\, d\rho}_{(\mathrm{I})}$$

Consider (I):

$$\frac{1}{2r}\left(\int_0^{r+t} \rho\, \vec{u}_t(x,0;\rho)\,d\rho + \underbrace{\int_{r-t}^{0} \rho\, \vec{u}_t(x,0;\rho)\,d\rho}_{}\right)$$

$$\parallel$$

$$-\int_0^{t-r} (-)\rho\, \vec{u}_t(x,0;\rho)(-1)\,d\rho \quad \text{for } \vec{u} \text{ is odd extention.}$$

$$= \frac{1}{2r}\int_{t-r}^{t+r} \rho\, \vec{u}_t(x,0;\rho)\,d\rho$$

Take lim of r:

$$\lim_{r\to 0} \frac{1}{2r}\int_{t-r}^{t+r} \rho\cdot \vec{u}_t(x,0;\rho)\,d\rho = t\cdot \vec{u}_t(x,0;t)$$

$$= t\cdot \int_{\partial B_t(x)} u_t\, ds \Big/ 4\pi t^2$$

$$= \frac{1}{4\pi t}\cdot \int_{\partial B_t(x)} u_t(\ ,0;t)\,ds.$$

Conside (II) :

Take lim of r,

$$-(-r+t)\cdot \vec{u}(\vec{x},0;-r+t)$$

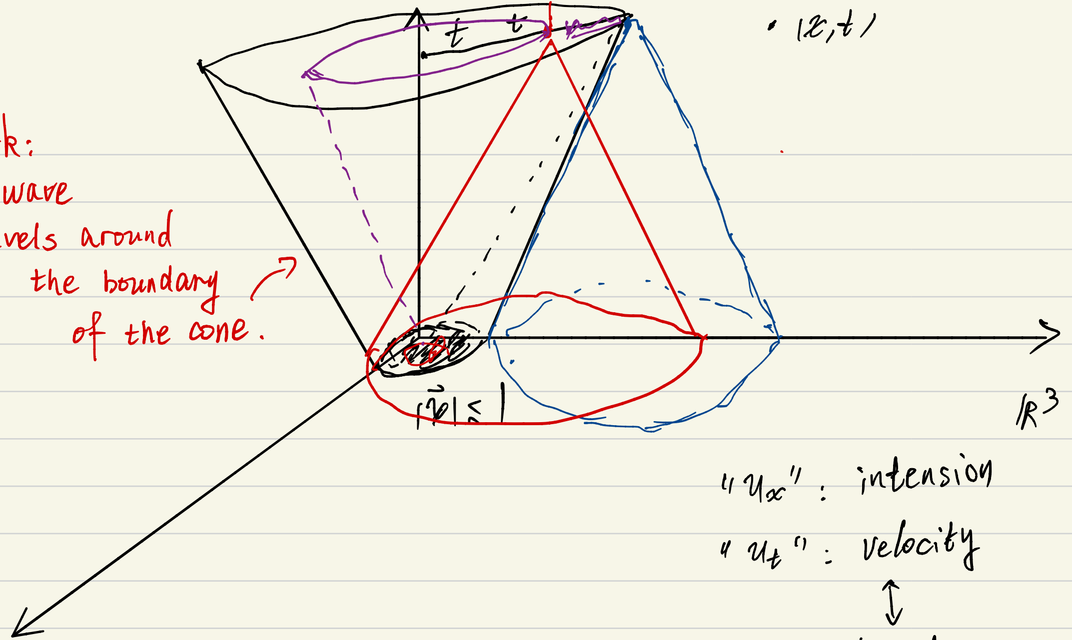
$$\lim_{r\to 0} \frac{(r+t)\cdot \vec{u}(\vec{x},0;r+t) + (r-t)\vec{u}(\vec{x},0;r-t)}{2r}$$

$$= \frac{\partial}{\partial t}\, t\,\vec{u}(x,0;t) = \vec{u}(x,0;t) + t\cdot \vec{u}_t(x,0;t)$$

$$u(\vec{x},t) = \frac{1}{4\pi t}\cdot \int_{\partial B_t(x)} u_t(\ ,0;t)\,ds \ +$$

$$\frac{\partial}{\partial t}\cdot \frac{1}{4\pi t}\int_{\partial B_t(x)} u(\ ,0;t)\,ds$$

Kirchhoff's formular.



Remark:

- The wave travels around the boundary of the cone.

"u_x" : intension

"u_t" : velocity

$\updownarrow$

kinetic energy

Q: What about wave eq. in 2-dimal. space?

A: The method doesn't work.

By same Poisson mean,

$$\Rightarrow \quad \frac{1}{r}\left(\frac{\partial \bar{u}}{\partial r} + r \cdot \frac{\partial^2 \bar{u}}{\partial r^2} \right) = \bar{u}_{tt}$$

This equation can't reduce to be a wave equation in 1-D. !

Another way to solve it: (See 2-diml. to be embeded in 3-diml.)

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x, y \in \mathbb{R},\ t > 0 \\ u(x, y, 0) = u_0(x, y), & u_t(x, y, 0) = u_1(x, y) \end{cases}$$

Define $\bar{U}(x, y, z, t) = u(x, y, t)$,

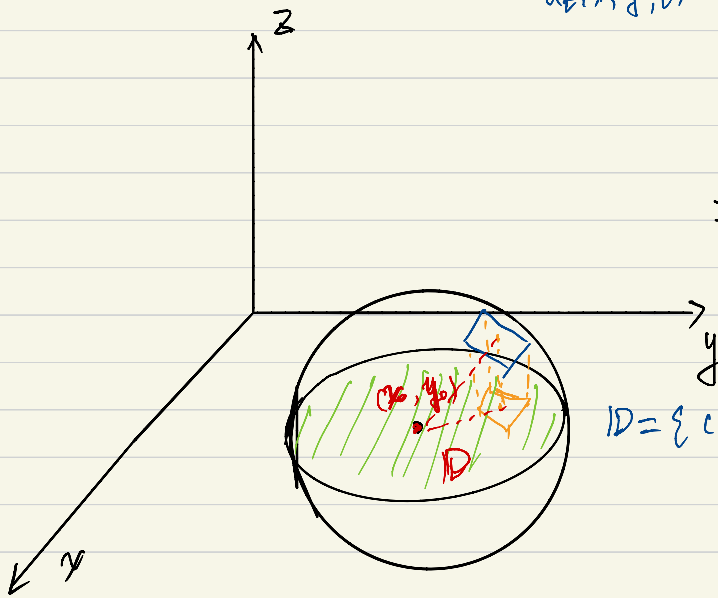
then $\bar{U}_{tt}(x, y, z, t) = u_{tt}(x, y, z)$

$$\bar{U}_{xx} + \bar{U}_{yy} + \bar{U}_{zz} = u_{xx} + u_{yy} + 0$$

Then,

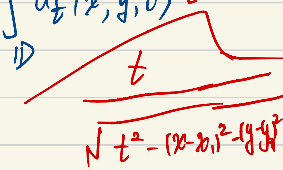
$$\underbrace{\bar{U}(\vec{x}, t)}_{=\,u(x,y,t)} = \frac{1}{4\pi t}\int_{\partial B_t(\vec{x})} \underbrace{\bar{U}_t(\vec{y},0)}_{=\,u_t(x,y,0)}\, dS_{\vec{y}} + \frac{\partial}{\partial t}\frac{1}{4\pi t}\int_{\partial B_t(\vec{x})} \bar{U}(\vec{y},0)\, dS_{\vec{y}}$$

$\vec{x} = (x, y, z)$



$$\frac{1}{4\pi t}\cdot\int_{\partial B_t(\vec{x})} \bar{U}_t(\vec{y},0)\, dS_{\vec{y}}$$

$$= \frac{2}{4\pi t}\int_D u_t(x,y,0)\cdot$$

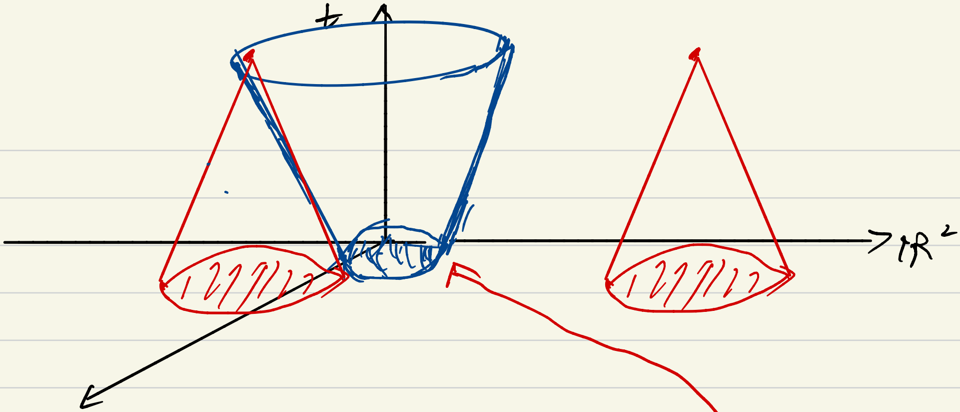


$D = \{(x_1, y_1) \mid |(x,y) - (x_1, y_1)| \le t\}$

$$\frac{1}{4\pi t}\cdot\int_{\partial B_t(\vec{x})} \bar{U}_t(\vec{y},0)\, dS_{\vec{y}} = \frac{1}{2\pi}\iint_{\sqrt{(x_1-x)^2+(y_1-y)^2}\le t} \frac{u_1(x_1,y_1)}{\sqrt{t^2-(x-x_1)^2-(y-y_1)^2}}\, dx_1\, dy_1$$

Take $\vec{x} = (x, y)$

$$u(\vec{x}, t) = \frac{1}{2\pi}\iint_{|\vec{y}-\vec{x}|\le t} \frac{u_1(\vec{y})}{\sqrt{t^2-|\vec{x}-\vec{y}|^2}}\, d\vec{y} + \frac{\partial}{\partial t}\frac{1}{2\pi}\iint_{|\vec{y}-\vec{x}|\le t} \frac{u_0(\vec{y})}{\sqrt{t^2-|\vec{x}-\vec{y}|^2}}\, d\vec{y}$$



Remark: • The wave travels inside of the cone!
Big difference with 3-dim.

Now, define

$$\mathcal{F}(u_1) = \frac{1}{2\pi} \iint_{|\vec{x}-\vec{y}|\le t} \frac{u_1(\vec{y})}{\sqrt{t^2 - (\vec{x}-\vec{y})^2}} \, d\vec{y}$$

$$\mathcal{F}(u_1) \equiv \text{solution of } \begin{cases} u_{tt} - u_{xx} - u_{yy} = 0 \\ u(x,y,0) = 0 \\ u_t(x,y,0) = u_1 \end{cases}$$

By Fourier Transform.

$$\Rightarrow \hat{u}(\vec{\eta}, t) = \frac{e^{i|\vec{\eta}|t} - e^{-i|\vec{\eta}|t}}{2i|\vec{\eta}|} \hat{u}_1(\vec{\eta}) = \frac{\sin |\vec{\eta}|t}{|\vec{\eta}|} \hat{u}_1(\vec{\eta})$$

$$\Updownarrow \leftarrow \text{Must be equivalent}$$

$$\mathcal{F}(u_1)$$

Fermat Last Thm:

$x^n + y^n = z^n$, $n > 2$

Half Space:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x>0,\ y\in\mathbb{R},\ t>0 \\ u(0,y,t) = 0 \\ \cancel{u(x,y,0) = u_0(x,y) = 0} \\ u_t(x,y,0) = u_1(x,y) = \delta(x-x_*)\,\delta(y) \end{cases}$$

Laplace – Fourier Transform.

Define

$$\mathcal{L}u(x,\eta,s) = \int_0^\infty\int_{\mathbb{R}} e^{-st-i\eta y}\, u(x,y,t)\, dy\, dt$$

$$\Rightarrow \quad s^2\mathcal{L}u - \hat{u}_1(x,\eta) - \partial_x^2\mathcal{L}u + \eta^2\mathcal{L}u = 0$$

$$\mathcal{L}u(0,\eta,s) = 0$$

$$\Rightarrow \begin{cases} (s^2+\eta^2)\mathcal{L}u - \partial_x^2\mathcal{L}u = \hat{u}_1 = \delta(x-x_*) \\ \mathcal{L}u(0,\eta,s) = 0 \end{cases} \qquad (*)$$

We already have known:

$$\begin{cases} w_{tt} - w_{xx} - w_{yy} = 0, & x, y\in\mathbb{R} \\ w(x,y,0) = 0 \\ w_t(x,y,0) = \delta(x)\cdot\delta(y) \end{cases}$$

$$\Rightarrow \quad (s^2+\eta^2)\mathcal{L}w - \partial_x^2\mathcal{L}w = \delta(x)$$

$$\mathcal{L}w = \frac{e^{-\sqrt{s^2+\eta^2}\,|x|}}{2\sqrt{s^2+\eta^2}}$$

Then, by $(*)$, we get:

$$\mathcal{L}w = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_*|}}{2\cdot\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+\eta^2}\,x}$$

$$\Rightarrow A = -\frac{e^{-\sqrt{s^2+\eta^2}\cdot|x_*|}}{2\cdot\sqrt{s^2+\eta^2}}$$

$$\Rightarrow \mathcal{L}u = \frac{e^{-\sqrt{s^2+\eta^2}\cdot|x-x_*|}}{2\sqrt{s^2+\eta^2}} - \frac{e^{-\sqrt{s^2+\eta^2}\cdot(x+x_*)}}{2\cdot\sqrt{s^2+\eta^2}}, \quad x_*>0$$

!

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x>0,\ y\in\mathbb{R} \\ u(x,y,0)=0, & u_t(x,y,0)=\delta(x-x_0)\,\delta(y) \\ u(0,y,t) - k\, u_x(0,y,t) = 0. \end{cases}$$

Transform in (t, y).

Take Laplace - Fourier Transform,

$$\begin{cases} s^2 \mathcal{L}u - \partial_x^2 \mathcal{L}u + \eta^2 \mathcal{L}u = \delta(x-x_0) \\ \mathcal{L}u(0,\eta,s) - k\,\partial_x \mathcal{L}u(0,\eta,s) = 0 \end{cases} \text{ — B.C.}$$

$$\Rightarrow \mathcal{L}u(x,\eta,s) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+\eta^2}\,x}$$

Use B.C.,

$$\Rightarrow \frac{e^{-\sqrt{s^2+\eta^2}\,|x_0|}}{2\sqrt{s^2+\eta^2}} + A - k\left(\frac{e^{\sqrt{s^2+\eta^2}\,(x-x_0)}}{2\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+t^2}\,x}\right)_x\Bigg|_{x=0} = 0$$

$$\Downarrow$$

$$\sqrt{s^2+\eta^2}\left(\frac{e^{-\sqrt{s^2+\eta^2}\,x_0}}{2\sqrt{s^2+\eta^2}} - A\right)$$

$$\Rightarrow (1+k\sqrt{s^2+\eta^2})\,A + (1-k\sqrt{s^2+\eta^2})\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,x_0}}{2\sqrt{s^2+\eta^2}} = 0$$

$$\Rightarrow A = \frac{-e^{-\sqrt{s^2+\eta^2}\,x_0}\,(1-k\sqrt{s^2+\eta^2})}{2\sqrt{s^2+\eta^2}\cdot(1+k\sqrt{s^2+\eta^2})}$$

$$\Rightarrow \mathcal{L}u(x,\eta,s) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1-k\sqrt{s^2+\eta^2})}{(1+k\sqrt{s^2+\eta^2})}\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$\times\ (1-k\sqrt{s^2+\eta^2})$

$$\Downarrow$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1-k\sqrt{s^2+\eta^2})^2}{[1-k^2(s^2+\eta^2)]} \cdot \frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1+d_x)^2}{[1-k^2(s^2+\eta^2)]} \cdot \frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

Inverse Laplace Transform:

$f(t)$: $\quad F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt, \quad \mathrm{Re}(s) \geq 0, \quad s \in \mathbb{C}$

$$f(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=0} e^{st} \cdot F(s)\, ds$$

$$\mathcal{L}^{-1}\left[\frac{1}{(1-k^2(s^2+\eta^2))}\right] = \frac{1}{4\pi^2 i} \int_{-\infty}^{\infty}\int_{\mathrm{Re}(s)=0} e^{i\eta y + st} \cdot \frac{1}{[1-k^2(s^2+\eta^2)]}\, ds\, d\eta$$

Another boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, \quad x>0, \ y \in \mathbb{R} \\ u(x,y,0) = 0, \quad u_t(x,y,0) = \delta(x-x_0)\,\delta(y) \\ u_t(0,y,t) - K\, u_x(0,y,t) = 0. \qquad \text{(B.C.)} \end{cases}$$

B.C. $\Rightarrow$ $s\,\mathcal{L}u = K\,\partial_x \mathcal{L}u.$

$$s\,\mathcal{L}u(x,\eta,s) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+\eta^2}\,x}$$

By the process above, change "1" to "s":

$$\Rightarrow (s + K\sqrt{s^2+\eta^2})\,A + \frac{e^{-\sqrt{s^2+\eta^2}\,x_0}}{2\sqrt{s^2+\eta^2}}\,(s - K\sqrt{s^2+\eta^2}) = 0$$

$$\mathcal{L}u(x,y,t) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(s-K\sqrt{s^2+\eta^2})^2}{[s^2-K^2(s^2+\eta^2)]}\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$$\Downarrow$$

$$\frac{(s^2 - 2K\sqrt{s^2+\eta^2}\,s + K^2(s^2+\eta^2))}{[s^2-K^2(s^2+\eta^2)]}$$

$$\Downarrow$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{[s^2+2Ks\,\partial_x + K^2\partial_x^2]}{(s^2-K^2(s^2+\eta^2))}\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\cdot\sqrt{s^2+\eta^2}} - [s^2+2Ks\,\partial_x + K^2\partial_x^2]\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\cdot\sqrt{s^2+\eta^2}}\cdot\frac{1}{(1-K^2)s^2-K^2\eta^2}$$

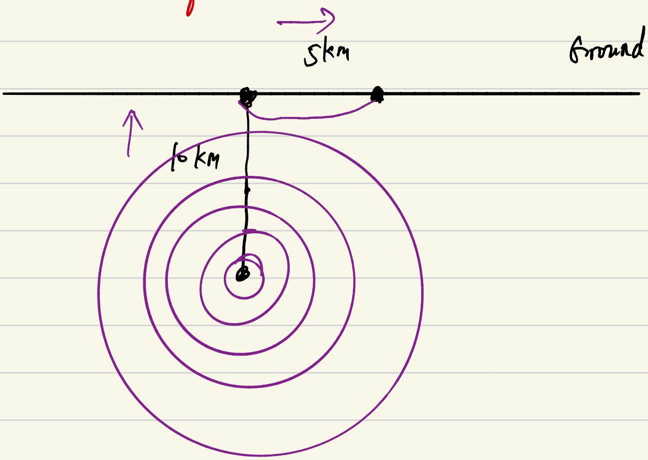
$$\rightsquigarrow -\frac{1}{k^2}\mathcal{L}\left[\left(\frac{k^2-1}{k^2}\partial_t^2-\partial_y^2\right)^{-1}\right]$$

$$\frac{-1/k^2}{-[(1-k^2)s^2-k^2\eta^2]/k^2} = \frac{-1/k^2}{\frac{k^2-1}{k^2}s^2+\eta^2}$$

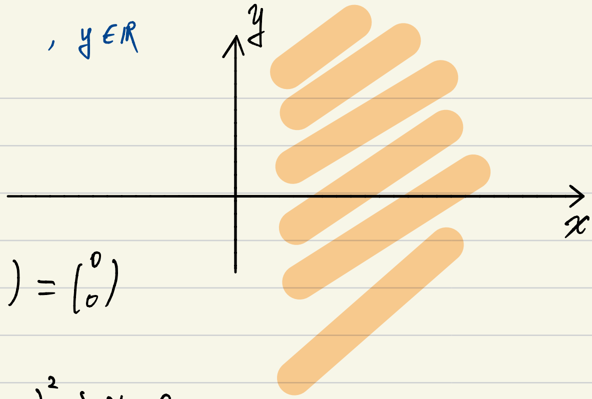
Remark: ① speed: $\frac{k^2}{k^2-1}$

② If k small, then no wave eq.

Physical meaning:



$$\begin{cases} u_t - u_x + v_y = u_{xx} + u_{yy}, & x>0 \\ v_t + v_x + u_y = v_{xx} + v_{yy}, & y \in \mathbb{R} \end{cases}$$



i.e. $\begin{cases} u_t - u_x + v_y = \Delta u \\ v_t + v_x + u_y = \Delta v \end{cases}$

$$\begin{pmatrix} \partial_t - \partial_x - \Delta & \partial_y \\ \partial_y & \partial_t + \partial_x - \Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} ((\partial_t - \partial_x - \Delta)\cdot(\partial_t + \partial_x - \Delta) - \partial_y^2)\, u = 0 \\ ((\partial_t - \partial_x - \Delta)\cdot(\partial_t + \partial_x - \Delta) - \partial_y^2)\, v = 0 \end{cases}$$

$$\Rightarrow \left[(\partial_t - \Delta)^2 - \partial_x^2 - \partial_y^2\right] u = 0 \quad \&$$

$$[(\partial_t - \Delta)^2 - \Delta]\cdot u = 0$$

Suppose $[(\partial_t - \Delta)^2 - \Delta]\, u = 0$ is an Equation on $\mathbb{R}^2 \times \mathbb{R}_+$

$\hat{u}(\vec{\eta}, t)$: Fourier transform of $u(x, y, t)$, $\vec{\eta} \in \mathbb{R}^2$

$$\Rightarrow [(\partial_t + |\vec{\eta}|^2)^2 + |\vec{\eta}|^2]\, \hat{u} = 0$$

$$\Rightarrow (\partial_t + |\vec{\eta}|^2 - i|\vec{\eta}|)(\partial_t + |\vec{\eta}|^2 + i\cdot|\vec{\eta}|)\, \hat{u} = 0$$

Then, $\hat{u}(\vec{\eta}, t) = A\cdot e^{(-|\vec{\eta}|^2 + i|\vec{\eta}|)t} + B\cdot e^{-|\vec{\eta}|^2 t - i|\vec{\eta}|t}$

$$= e^{-|\vec{\eta}|^2 t}\cdot(A\cdot e^{i|\vec{\eta}|t} + B\cdot e^{-i|\vec{\eta}|t})$$

$$\overset{?}{=} e^{-|\vec{\eta}|^2 t}\cdot\left(A\cdot \frac{\sin|\vec{\eta}|t}{|\vec{\eta}|} + B\cdot\cos|\vec{\eta}|t\right)$$

$A = \hat{u}_t(\vec{\eta}, 0)$

$B = \hat{u}(\vec{\eta}, 0)$

Fourier - Laplace transform:

$$\mathcal{L}\left[\begin{pmatrix} \partial_t - \partial_x - \Delta & \partial_y \\ \partial_y & \partial_t + \partial_x - \Delta \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \hat{u}(x,\eta,0) \\ \hat{v}(x,\eta,0) \end{bmatrix}$$

Suppose $\begin{bmatrix} u(x,y,0) \\ v(x,y,0) \end{bmatrix} = \begin{bmatrix} \delta(x)\,\delta(y) \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \delta(x)\cdot\delta(y) \end{bmatrix}$.

If $\begin{bmatrix} u(x,y,0) \\ v(x,y,0) \end{bmatrix} = \begin{bmatrix} \delta(x)\,\delta(y) \\ 0 \end{bmatrix}$,

$$\begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} \mathcal{L}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \delta(x) \\ 0 \end{bmatrix}$$

Consider

$$\begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} \Psi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x > 0$$

Assume $\Psi = e^{\lambda x} \cdot \vec{V}_0$ is a solution

$$\Rightarrow \begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} e^{\lambda x} \cdot \vec{V}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s - \lambda - \lambda^2 + \eta^2 & i\eta \\ i\eta & s + \lambda - \lambda^2 + \eta^2 \end{bmatrix} \vec{V}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad (s+\eta^2-\lambda^2)^2-\lambda^2+\eta^2=0$$

$$V_0 = \begin{bmatrix} -i\eta \\ s-\lambda-\lambda^2+\eta^2 \end{bmatrix}$$

$$\alpha = \lambda^2 \qquad \Rightarrow \quad (s+\eta^2-\alpha)^2-\alpha+\eta^2=0$$

$$\Rightarrow \quad \alpha^2-2(s+\eta^2)\alpha-\alpha+(s+\eta^2)^2+\eta^2=0$$

$$\Rightarrow \quad \alpha = \frac{2(s+\eta^2)+1 \pm \sqrt{[2(s+\eta^2)+1]^2-4[(s+\eta^2)^2+\eta^2]}}{2}$$

Since $[2(s+\eta^2)+1]^2-4[(s+\eta^2)^2+\eta^2]$

$= 4s+1$,

then $\quad \alpha = \dfrac{2(s+\eta^2)+1 \pm \sqrt{4s+1}}{2}$

$$\lambda = \pm\sqrt{\frac{2(s+\eta^2)+1 \pm \sqrt{4s+1}}{2}}$$

$$\Rightarrow \quad \lambda^R_{\pm} = -\sqrt{\frac{2(s+\eta^2)+1 \pm \sqrt{4s+1}}{2}}$$

&

$$\lambda^L_{\pm} = \sqrt{\frac{2(s+\eta^2)+1 \pm \sqrt{4s+1}}{2}}$$

- $V_0 = \begin{bmatrix} -i\eta \\ s-\lambda-\lambda^2+\eta^2 \end{bmatrix}$,

$$e^{\lambda^L_{\pm} x} \begin{pmatrix} -i\eta \\ s-\lambda^L_{\pm}-\lambda^{L\,2}_{\pm}+\eta^2 \end{pmatrix}$$

$$e^{\lambda^R_{\pm} \cdot x} \cdot \begin{pmatrix} -i\eta \\ s-\lambda^R_{\pm}-\lambda^{R\,2}_{\pm}+\eta^2 \end{pmatrix}$$

$$Q(x) = \begin{cases} L_+ \cdot e^{\lambda_+^L x} \cdot \left(\dfrac{-i\eta}{s - \lambda_+^L - (\lambda_+^L)^2 + \eta^2} \right) + L_- e^{\lambda_-^L \cdot x} \left[\dfrac{-i\eta}{s - \lambda_-^L - (\lambda_-^L)^2 + \eta^2} \right], & x > 0 \\ R_+ \cdot e^{\lambda_+^R x} \cdot \left[\dfrac{-i\eta}{s - \lambda_+^R - (\lambda_+^R)^2 + \eta^2} \right] + R_- \cdot e^{\lambda_-^R \cdot x} \cdot \left[\dfrac{-i\eta}{s - \lambda_-^R - (\lambda_-^R)^2 + \eta^2} \right], & x < 0 \end{cases}$$

Continuity:

$$L_+ \left(\frac{-i\eta}{s - \lambda_+^L - (\lambda_+^L)^2 + \eta^2} \right) + L_- \cdot \left(\frac{-i\eta}{s - \lambda_-^L - (\lambda_-^L)^2 + \eta^2} \right)$$

$$= R_+ \left[\frac{-i\eta}{s - \lambda_+^R - (\lambda_+^R)^2 + \eta^2} \right] + R_- \left[\frac{-i\eta}{s - \lambda_-^R - (\lambda_-^R)^2 + \eta^2} \right]$$

- Consider $A(\eta, s) \cdot e^{-\sqrt{\frac{2(s+\eta^2)+1 \pm \sqrt{4s+1}}{2}}\, x}$, $x > 0$

- How to compute inverse of $A(\eta, s) \cdot e^{-\sqrt{2(s+\eta^2)+1 \pm \sqrt{4s+1}}\, x}$?

$$\sqrt{\frac{2(s+\eta^2)+1-2\sqrt{4s+1}}{2}} = \sqrt{2(s+\eta^2) - 2\sqrt{s+1/4} + 1 \;/2}$$

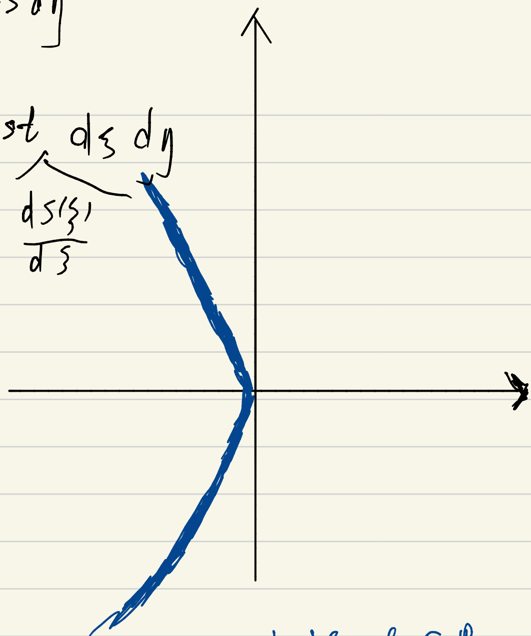
$$= \sqrt{s + \tfrac{1}{4} + \tfrac{1}{4} - \sqrt{s+1/4} + \eta^2} = \sqrt{(\sqrt{s+1/4} - \tfrac{1}{2})^2 + \eta^2} = \lambda.$$

$$\Rightarrow \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{Re(s)=0} \int_{\mathbb{R}} e^{-\lambda x + i\eta y + st} \cdot d\eta\, ds$$

Fourier-Laplace Path:

$$\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{\mathrm{Re}(s)=0} e^{-\lambda x+i\eta y+st}\,ds\,d\eta$$

$$=\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{-\infty}^{\infty} e^{i\xi x+i\eta y+st}\,\underbrace{d\xi}_{\frac{ds(\xi)}{d\xi}}\,d\eta$$



$\lambda(s(\xi))=i\xi,\ \xi\in\mathbb{R}$

Want : $\lambda(s(\xi)) = i\xi,\ \xi\in\mathbb{R}$

$$\sqrt{\left(\sqrt{s+\tfrac{1}{4}}-\tfrac{1}{2}\right)^2+\eta^2} = -i\xi,$$

then,

$$\left(\sqrt{s+1/4}-\tfrac{1}{2}\right)^2+\eta^2 = -\xi^2$$

$$\Rightarrow \left(\sqrt{s+1/4}-1/2\right)^2 = -\xi^2-\eta^2$$

$$\Rightarrow \sqrt{s+1/4}-1/2 = \pm i\sqrt{\eta^2+\xi^2}$$

$$\left(\sqrt{s+1/4}\right)^2 = \left(\pm i\sqrt{\eta^2+\xi^2}+\tfrac{1}{2}\right)^2$$

$$\Rightarrow \underline{s = -(\eta^2+\xi^2)\pm i\sqrt{\eta^2+\xi^2}}$$

Thus,

$$\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{-\infty}^{\infty} e^{i\xi x+i\eta y\underbrace{-(\eta^2+\xi^2)t}_{\text{heat eq.}}\underbrace{\pm i\sqrt{\eta^2+\xi^2}\,t}_{\text{wave eq.}}}\cdot\frac{ds(\xi)}{d\xi}\,d\xi\,d\eta$$

$$\frac{ds}{d\xi} = -2\xi\pm i\frac{\xi}{\sqrt{\eta^2+\xi^2}}$$

$$=\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{-\infty}^{\infty} e^{i\xi x+i\eta y-(\eta^2+\xi^2)t\pm i\sqrt{\eta^2+\xi^2}\,t}\cdot\left(-2\xi\pm\frac{i\xi}{\sqrt{\eta^2+\xi^2}}\right)d\xi\,d\eta$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \cdot (-2\xi) \, d\xi \, d\eta$$

$$+ \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \cdot \left(\frac{\pm i \xi}{\sqrt{\eta^2+\xi^2}} \right) d\xi \, d\eta$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} 2i \, \partial_x e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \, d\xi \, d\eta$$

$$+ \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} \underbrace{\partial_x e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \cdot \frac{\pm 1}{\sqrt{\eta^2+\xi^2}}}_{\parallel} \, d\xi \, d\eta$$

$$\underbrace{\partial_x \cdot e^{i\xi x + i\eta y} \cdot e^{-(\eta^2+\xi^2)t} \cdot \frac{e^{\pm i\sqrt{\eta^2+\xi^2}\, t}}{\pm\sqrt{\eta^2+\xi^2}}}$$

$$\Downarrow$$

$$\partial_x \mathcal{F}^{-1}\left(e^{-(\eta^2+\xi^2)t}\right) * \mathcal{F}^{-1}\left(\frac{e^{\pm i\sqrt{\eta^2+\xi^2}\, t}}{\pm\sqrt{\eta^2+\xi^2}} \right)$$

$$\downarrow$$

$$\mathcal{F}^{-1}\left(\frac{\sin \sqrt{\eta^2+\xi^2}\, t}{\sqrt{\eta^2+\xi^2}} \right)$$

Q.E.D.

Navier-Stokes Eq.:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (um)_x + P(\rho)_x = u_{xx}, \end{cases}$$

m: momentum
ρ: density of fluid
u: fluid velocity
$P(\rho)$: Pressure
$P(\rho) = \rho^\gamma,\ \gamma \in (1, 5/3)$

A Linearized Eq.

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \rho_x = m_{xx} \end{cases}$$

① $\begin{cases} \rho_t + m_x = 0 \\ m_t + \rho_x = 0 \end{cases}$ wave eq.

$$\Downarrow$$

$$(\rho_t + m_x)_x = 0 = (m_t + \rho_x)_t$$

$$\Downarrow$$

$$m_{xx} - m_{tt} = 0.$$

② $\begin{pmatrix} \partial_t & \partial_x \\ \partial_x & \partial_t \end{pmatrix} \begin{pmatrix} \rho \\ m \end{pmatrix} = 0.$

$$\Rightarrow \quad \partial_{tt} - \partial_{xx} = 0.$$

③ $\begin{cases} \rho_t + m_x = \rho_{xx} \\ m_t + \rho_x = m_{xx} \end{cases} \Rightarrow \begin{pmatrix} \partial_t - \partial_x^2 & \partial_x \\ \partial_x & \partial_t - \partial_x^2 \end{pmatrix} \begin{pmatrix} \rho \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$[(\partial_t - \partial_x^2)^2 - \partial_x^2]\, \rho = 0$$

$$[(\partial_t - \partial_x^2)^2 - \partial_x^2]\, m = 0$$

By Fourier Transform

$\Downarrow$

$$\left[(\partial_t + \eta^2)^2 + \eta^2 \right] \hat{\rho} = 0$$

$\Downarrow$

$$\hat{\rho} = A_+ e^{\underbrace{-\eta^2 t}_{\text{heat eq.}} \underbrace{+ i\eta t}_{\text{transport.}}} + A_- e^{-\eta^2 t - i\eta t}$$

Weak solution: Nonlinear Problem:

$$u_t + f(u)_x = 0$$

$$\Rightarrow \iint \varphi(x,t) \cdot (u_t + f(u)_x)\, dx\, dt = 0, \quad \varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$$

$$\Downarrow$$

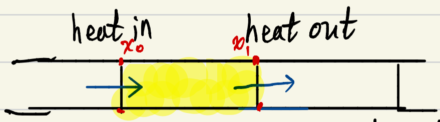
$$(*) \quad \iint [-\varphi_t(x,t) \cdot u(x,t) - \varphi_x(x,t) f(u)]\, dx\, dt = 0,$$

A solution u satisfying $(*)$ is called weak sol.

Heat Eq:

$$u_t = u_{xx}$$

Fourier's Law: Heat flux is proportion to temperature gradient.



$$u_t \propto \text{heat in} - \text{heat out}$$

- Flux: u at x_0 $\sim -k\, u_x(x_0, t)$

$$u_t = [-k(x_0) \cdot u_x(x_0,t) + k(x_1) \cdot u_x(x_1,t)]$$

$$\Rightarrow u_t = \lim_{x_1 \to x_0} \frac{[-k(x_0) \cdot u_x(x_0,t) + k(x_1) \cdot u_x(x_1,t)]}{x_1 - x_0}$$

$$\Rightarrow u_t = \partial_x (k(x) u_x)$$

Consider weak solution: $\iint \varphi(u_t - \partial_x(k(x) \cdot u_x))\, dx\, dt = 0,$

$$\Rightarrow \quad \iint -\varphi_t u + \varphi_x \cdot k(x) \cdot u_x \, dx \, dt = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$$

u_x local integrable $\Rightarrow$ u: continuous.

Condition for $k(x)$:

$k(x)$: B.V. function :

Bounded Variation :
- Piecewise continuous
- Jump is countable

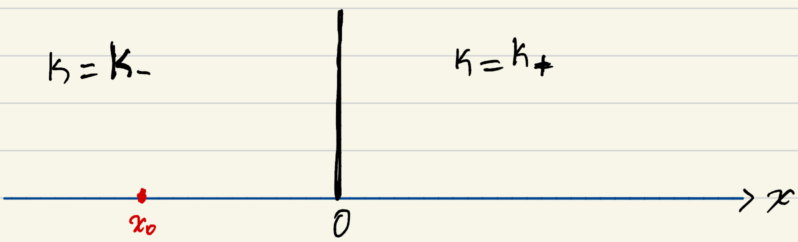
Step 1: k is const. & $k > 0$;

$$\begin{cases} u_t = k\, u_{xx} \\ u(x,0) = \delta(x) \end{cases} \xRightarrow{\text{Green Function.}} u(x,t) = \frac{e^{-\frac{x^2}{4k \cdot t}}}{\sqrt{4\pi t}}$$

$$\mathcal{L}\, u(x,s) = \int_0^\infty e^{-st} \cdot u(x,t) \, dt$$

$$\Rightarrow \mathcal{L}\, u(x,s) = \frac{e^{-\sqrt{s/k}\,|x|}}{2\sqrt{sk}}$$

Step 2:



$$\begin{cases} u_t = \partial_x (k(x) u_x) \\ u(x,0) = \delta(x - x_0), \quad x_0 < 0 \end{cases}$$

I.B.P.: $\iint \varphi y' \, dx = -\iint \varphi' y \, dx$

y: continous.

(i) $u(x)$: continuous.

(ii) $K(x)\, u_x$: contin.

$$\begin{cases} s\mathcal{L}u = \partial_x \left(K(x)\, \partial_x \mathcal{L}u \right) + \delta(x-x_0) \\ u(x,0) = \delta(x-x_0), \quad x_0 < 0 \end{cases}$$

$$\begin{cases} s\mathcal{L}u = \partial_x \left(K_- \, \partial_x \mathcal{L}u \right) + \delta(x-x_0) & \text{for } x<0 \\ s\mathcal{L}u = \partial_x \left(K_+ \, \partial_x \mathcal{L}u \right) & \text{for } x>0 \end{cases}$$

$$\Rightarrow \mathcal{L}u = \begin{cases} \dfrac{e^{-\sqrt{s/K_-}\cdot|x-x_0|}}{2\sqrt{sK_-}} + U_- \cdot e^{\sqrt{s/K_-}\,x}, & x<0 \\ S_+ \cdot e^{-\sqrt{s/K_+}\,x} & x>0 \end{cases}$$

By continuity of u at 0,

$$\frac{e^{-\sqrt{s/K_-}\cdot|x_0|}}{2\sqrt{sK_-}} + U_- = S_+ \qquad \text{--- (I)}$$

By continuity of flux:

$$\begin{cases} \left[\dfrac{e^{-\sqrt{s/K_-}\cdot|x-x_0|}}{2\sqrt{sK_-}} + U_- \cdot e^{\sqrt{s/K_-}\,x} \right]_x, & x<0 \\ \left(S_+ \cdot e^{-\sqrt{s/K_+}\,x} \right)_x & , \quad x>0 \end{cases}$$

$$\Rightarrow K_- \left(-\sqrt{\frac{s}{K_-}} \cdot \frac{e^{-\sqrt{s/K_-}\,|x_0|}}{2\sqrt{sK_-}} + U_- \sqrt{s/K_-} \right)$$

$$= K_+ \left(- S_+ \cdot \sqrt{\frac{s}{K_+}} \right) \qquad \text{--- (II)}$$

$$\Rightarrow \begin{cases} \dfrac{e^{\sqrt{s/k_-}\,|x_0|}}{2\sqrt{s\cdot k_-}} = \overset{\text{transmittion}}{S_+} - \overset{\text{reflexsion}}{U_-} \\[2ex] S_+ + \sqrt{\dfrac{k_-}{k_+}}\, U_- = \left(k_- \sqrt{\dfrac{1}{k_-}} \cdot \dfrac{e^{-\sqrt{s/k_-}\,|x_0|}}{2\sqrt{s k_-}} \right) \Big/ \sqrt{k_+} \end{cases}$$

Physical meaning:

1. If $k_- = k_+$, $U_- = 0$ i.e. no reflexsion

2. $U_- = O(1)\left(\sqrt{\frac{k_-}{k_+}} - 1\right) \cdot \dfrac{e^{-\sqrt{s/k_-}\,|x_0|}}{\sqrt{s k_-}}$

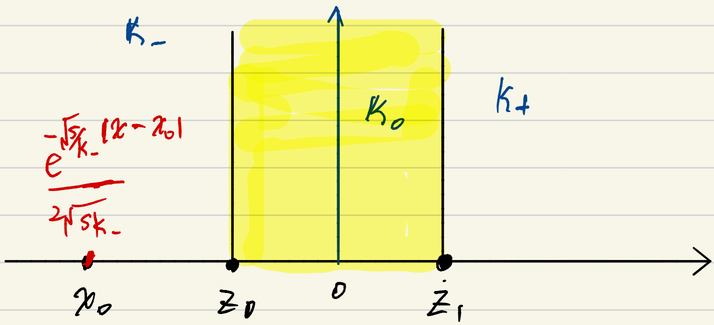
$$S_+ = \left[1 + O(1)\cdot\left(\sqrt{\frac{k_-}{k_+}} - 1\right)\right] \frac{e^{-\sqrt{s/k_-}\,|x_0|}}{\sqrt{s k_-}}$$

$$\Rightarrow \mathcal{L}u = \begin{cases} \dfrac{e^{-\sqrt{s/k_-}\cdot|x-x_0|}}{2\sqrt{s k_-}} + \underbrace{U_-\cdot e^{\sqrt{s/k_-}\,x}}_{\text{reflexsion}}, & x<0 \\[2ex] \underbrace{S_+\cdot e^{-\sqrt{s/k_+}\,x}}_{} \quad \text{transmittion.}, & x>0 \end{cases}$$

$$\Rightarrow \mathcal{L}_t u = \begin{cases} \dfrac{e^{-\sqrt{s/k_-}\cdot|x-x_0|}}{2\sqrt{s k_-}} + O(1)\left(\sqrt{\frac{k_-}{k_+}} - 1\right)\cdot\dfrac{e^{\sqrt{s/k_-}\,(x-|x_0|)}}{\sqrt{s k_-}}, & x<0 \\[2ex] \left[1 + O(1)\cdot\left(\sqrt{\frac{k_-}{k_+}} - 1\right)\right] \dfrac{e^{-\sqrt{s/k_-}\,|x_0| - \sqrt{s/k_+}\,x}}{\sqrt{s k_-}}, & x>0 \end{cases}$$

- $$\frac{e^{\sqrt{s/k_-}\,(x-|x_0|)}}{\sqrt{s k_-}} = \frac{e^{-\sqrt{s}\cdot\underbrace{\left(\frac{|x_0|}{\sqrt{k_-}} + \frac{|x|}{\sqrt{k_-}}\right)}_{\delta.}}}{\sqrt{s k_-}}$$

$$\Rightarrow \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}\cdot\sqrt{K_-}} = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}\cdot\sqrt{K_-}} = \frac{e^{-\left(\frac{x_0}{\sqrt{K_-}}+\frac{x}{\sqrt{K_-}}\right)^2}}{\sqrt{4\pi t K_-}}$$



• Observe:

$$\sqrt{\frac{K_-}{K_+}} - 1 = \sqrt{\frac{\underbrace{K_- - K_+}_{\text{small enough}}}{K_+} + 1} - 1$$

By Taylor' expansion,

$$\sqrt{\frac{K_- - K_+}{K_+} + 1} = \frac{(K_- - K_+)}{K_+} + 1 + \cdots$$

$$= O(1)\cdot(K_- - K_+).$$

$$\begin{cases} u_t - \partial_x(K(x)\,u_x) = 0 \\ u(x_0) = \delta(x - x_0) \end{cases}$$

Take Laplace transform,

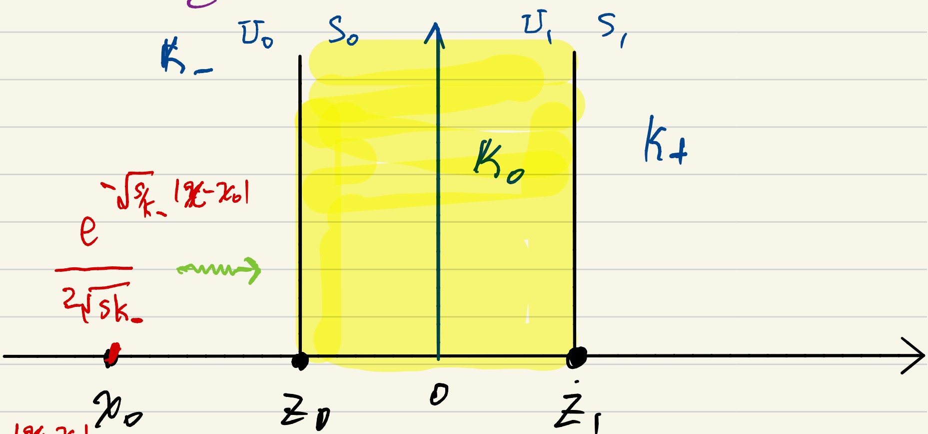
$$\Rightarrow s\mathcal{L}u - \partial_x(K(x)\,\partial_x \mathcal{L}u) = \delta(x - x_0)$$

$$(s - \partial_x K(x)\,\partial_x)\,\mathcal{L}u = \delta(x - x_0)$$

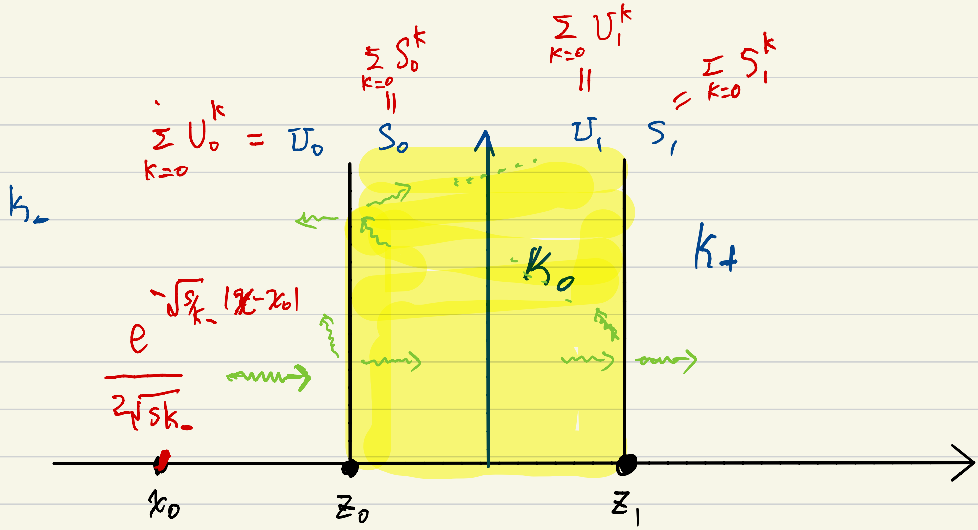
$$\Rightarrow \text{“}\mathcal{L}u = (s - \partial_x K(x)\,\partial_x)^{-1}\delta(x - x_0)\text{”}$$

No understanding! See the operator as spectrum, then by F.A.

Do it with understanding:



$$
\mathcal{L}u = \begin{cases} \dfrac{e^{-\sqrt{s/k_-}\,|x-x_0|}}{2\sqrt{sk_-}} + U_0 \cdot e^{\sqrt{s/k_-}\,x}, & x < z_0 \\ S_0 \cdot e^{-\sqrt{\frac{s}{k_-}}(x-z_0)} + U_1 \cdot e^{\sqrt{\frac{s}{k_0}}(x-z_1)}, & z_0 \le x \le z_1 \\ S_1 \cdot e^{-\sqrt{\frac{s}{k_0}}(x-z_1)}, & x \geq z_1 \end{cases}
$$



R^0_{++} T^0_{+-} $\qquad$ R^1_{++} T^1_{+-} all are const. about x.

R^0_{--} T^0_{-+} $\qquad$ R^1_{--} T^0_{-+}

$$U_0^0 = R^0_{--}\cdot\frac{e^{-\sqrt{s/k_-}\,|x_0|}}{2\cdot\sqrt{sk_-}}, \qquad S_0^0 = T^0_{-+}\,\frac{e^{-\sqrt{s/k_-}\,|x_0|}}{2\sqrt{sk_-}}$$

$$U_0^1 = 0, \qquad S_0^1 = 0$$

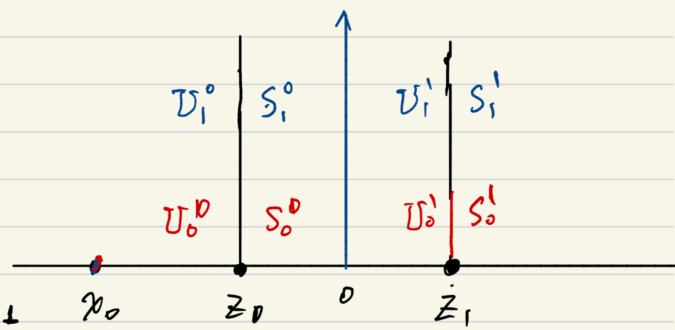
- $S_1^0 = U_0^1\cdot e^{\sqrt{s/k_0}(z_0-z_1)}\cdot R^0_{++}$

$U_1^0 = U_0^1\cdot e^{\sqrt{s/k_0}(z_0-z_1)}\cdot T^0_{+-}$

$U_1^1 = S_0^0\cdot e^{\sqrt{s/k_0}(z_0-z_1)} R^1_{--}$

$S_1^1 = S_0^0\cdot e^{\sqrt{s/k_0}(z_0-z_1)}\cdot T^1_{-+}$

$\vdots \quad \vdots \qquad \vdots$



$$S_k^0 = U_{k-1}^1 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot R_{++}^0$$

$$U_k^0 = U_{k-1}^1 e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot T_{+-}^0$$

$$U_k^1 = S_{k-1}^0 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} R_{--}^1$$

$$S_k^1 = S_{k-1}^0 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot T_{-+}^1$$

$$\vdots \qquad \vdots \qquad \vdots$$

Observe:

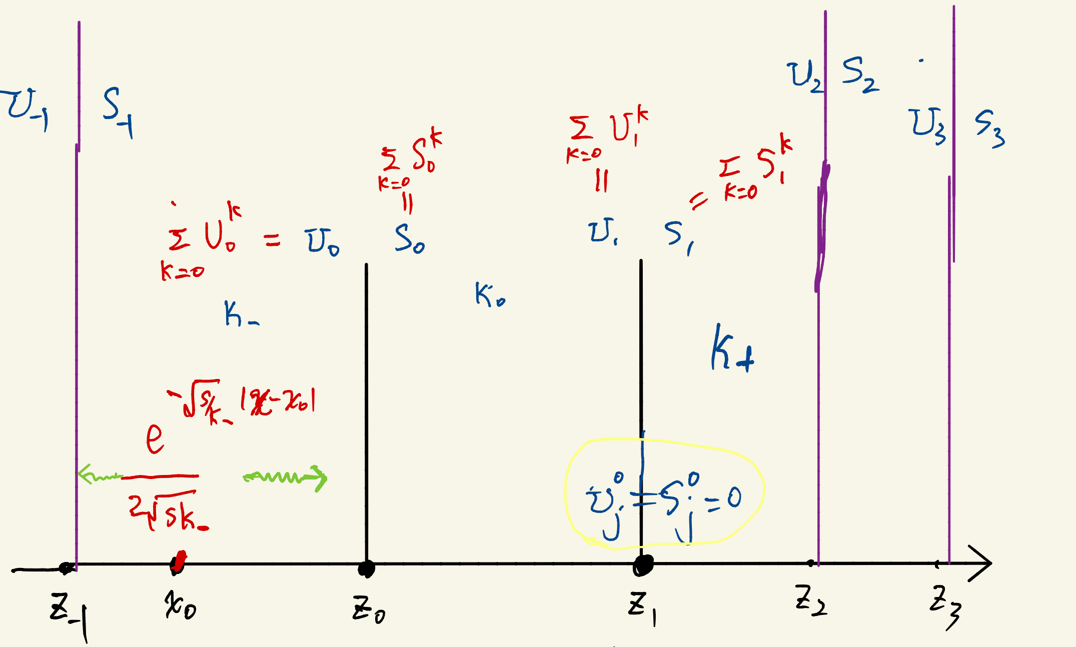
$$S_{2k}^0 = e^{2\sqrt{s/k_0}(z_0 - z_1)k} \cdot R_{--}^1 \cdot R_{++}^0 \cdot S_{2k-2}^0$$

$$\Rightarrow \quad S_{2k}^0 = e^{2\sqrt{s/k_0}(z_0 - z_1)k} \cdot (R_{--}^1 R_{++}^0)^k \cdot \frac{e^{-\sqrt{s/k_-}|x_0|}}{2\sqrt{s k_-}}$$

$$\sum_{k=0} S_{2k}^0 = \sum \underbrace{(R_{--}^1 R_{++}^0)^k}_{\text{depending on } k.} \cdot \frac{e^{2\sqrt{s/k_0}(z_0 - z_1)k - \sqrt{s/k_0}|x_0|}}{2\sqrt{s k_-}}$$

$$\frac{e^{2\sqrt{s/k_0}(z_0 - z)k - \sqrt{s/k_0}|x_0|}}{2\sqrt{s k_-}} = \frac{e^{\sqrt{s}\left(2\sqrt{1/k_0}(z_0 - z)k - \sqrt{1/k_0}|x_0|\right)}}{2\sqrt{s k_-}}$$

$$\longleftrightarrow \quad \frac{e^{-\frac{\left[2\sqrt{\frac{t}{k_0}}(z - z_0) - \sqrt{\frac{t}{k_0}}|x_0|\right]^2}{4t}}}{\sqrt{k_-} \cdot \sqrt{4\pi t}}.$$

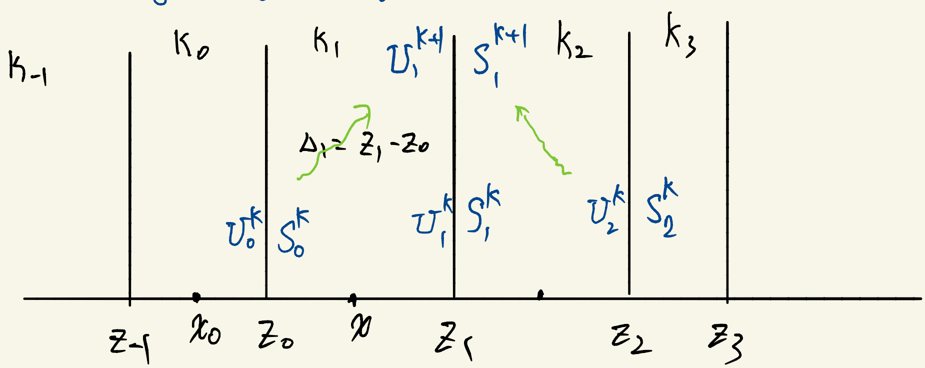


$$U_j = \sum_{k=0}^{\infty} U_j^k \quad \& \quad S_j = \sum_{k=0}^{\infty} S_j^k$$

• $U_j^0 = S_j^0 = 0$, if $j \geq 1$ or $j \leq -2$

$$S_0^0 = \frac{e^{-\sqrt{s/k_-}\,|z_0 - x_0|}}{2\sqrt{sk_-}} \cdot T_{1-+}^0, \quad U_0^0 = \frac{e^{-\sqrt{s/k_-}\,|z_0 - x_0|}}{2\sqrt{sk_-}} \cdot R_{--}^0$$

• Suppose (S_j^k, U_j^k), $\forall j$.



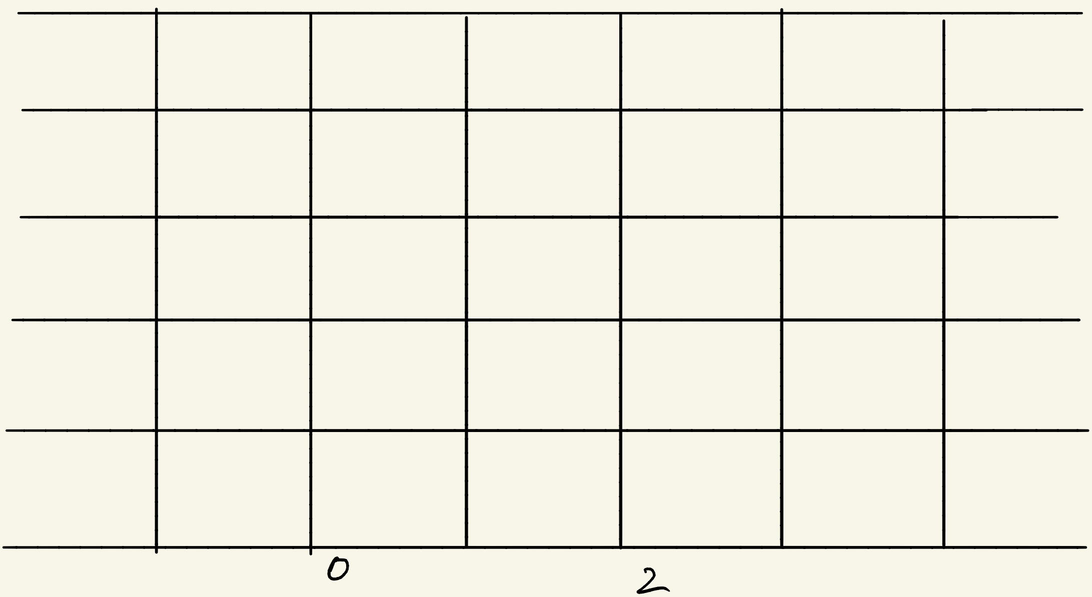
$$U_1^{k+1} = S_0^k \cdot e^{-\sqrt{s/K_1}\,\Delta_1} R^1_{--} + U_2^k \cdot e^{-\sqrt{s/K_2}\,\Delta_2} \cdot T^1_{+-}$$

&

$$S_1^{k+1} = S_0^k \cdot e^{-\sqrt{s/K_1}\,\Delta_1} T^1_{-+} + U_2^k \cdot e^{-\sqrt{s/K_2}\,\Delta_2} \cdot R^1_{++}$$

$$\Rightarrow U_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/K_j}\,\Delta_j} R^j_{--} + U_{j+1}^k \cdot e^{-\sqrt{s/K_{j+1}}\,\Delta_{j+1}} \cdot T^j_{+-}$$

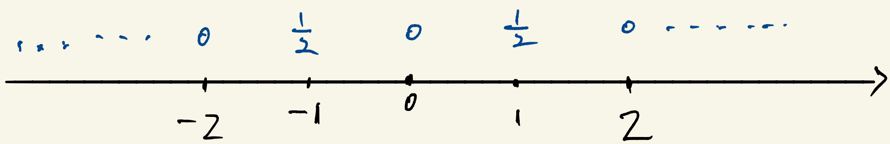
$$S_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/K_j}\,\Delta_j} T^j_{-+} + U_{j+1}^k \cdot e^{-\sqrt{s/K_{j+1}}\,\Delta_{j+1}} \cdot R^j_{++}$$

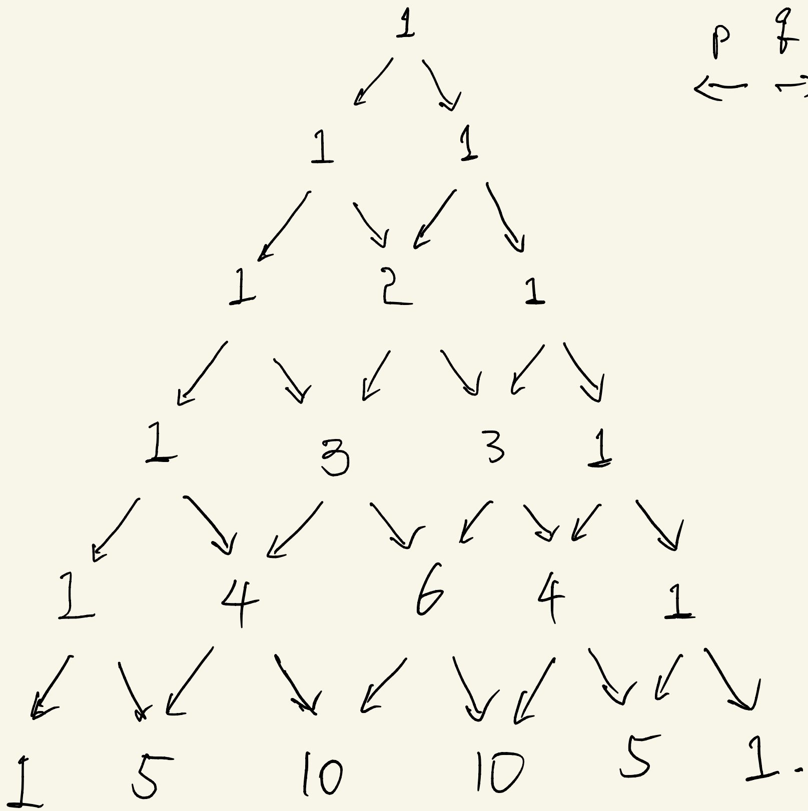


$\longleftarrow \longrightarrow$ $(n=6)$

$$\left(\frac{1}{2}x + \frac{1}{2}\frac{1}{x}\right)^6$$ Rondam Walk.

$$= \left(\frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{x}\right) \cdots\cdots \left(\frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{x}\right)$$





$\overset{p}{\longleftarrow} \quad \overset{q}{\longrightarrow}, \quad p+q=1$

Numerical Solution

$u_t + u_x = 0$

Numerical Sol.

$$u(j\Delta x, n\Delta t) = u_j^n$$

$$\Rightarrow \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$$\Rightarrow \quad u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x} \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

$$\Rightarrow \quad u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

$$= \underbrace{-\frac{\lambda}{2}} u_{j+1}^n + u_j^n + \frac{\lambda}{2} u_{j-1}^n$$

Need to be positive for using probility.

Another Scheme:

$$U_j^{n+1} - \frac{(u_{j+1}^n + u_j^n + u_{j-1}^n)}{3} + \frac{\Delta t}{\Delta x} \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

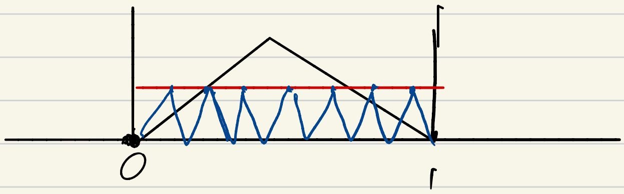
$\frac{\lambda}{2} < 1/3$

$$\Rightarrow \quad u_j^{n+1} = \left(\frac{1}{3} - \frac{\lambda}{2}\right) U_{j+1}^n + \frac{1}{3} U_j^n + \left(\frac{1}{3} + \frac{\lambda}{2}\right) U_{j-1}^n$$

B.V. : Bounded Variation Function

$$\sup_{\substack{P \\ \| \\ \{x_1, \dots\}}} \sum_i |K(x_i) - K(x_{i+1})| < \infty$$

Brown Motion



Now, assume $K(x)$ is B.V. function.

$$\|K\|_{B.V.} = \sup_{P} \sum_i |K(x_i) - K(x_{i+1})| < \infty$$

$P = \{x_1, \dots\}$.

- If K is continuous, then $\|K\|_{B.V} = \int_{\mathbb{R}} |K_x| \, dx$

$$\begin{cases} U_j^{k+1} = S_{j-1}^{k} \cdot e^{-\sqrt{S/K_j}\,\Delta_j} R_{--}^{j} + U_{j+1}^{k} \cdot e^{-\sqrt{S/K_{j+1}}\,\Delta_{j+1}} \cdot T_{+-}^{j} \\ S_j^{k+1} = S_{j-1}^{k} \cdot e^{-\sqrt{S/K_j}\,\Delta_j} T_{-+}^{j} + U_{j+1}^{k} \cdot e^{-\sqrt{S/K_{j+1}}\,\Delta_{j+1}} \cdot R_{++}^{j} \end{cases}$$

$$\begin{bmatrix} U_j^{k+1} \\ S_j^{k+1} \end{bmatrix} = R_{j-1} \begin{bmatrix} U_{j-1}^{k} \\ S_j^{k} \end{bmatrix} + \mathcal{L}_{j+1} \cdot \begin{bmatrix} U_{j+1}^{k} \\ S_{j+1}^{k} \end{bmatrix}$$

$$\Omega^k := \{ w^k(j) = \{ {-1 \atop 1} \quad \}$$

$$\Sigma_j(w^k) = \sum_{l=0}^{j} w^k(l)$$

$$\bigcup_{k=0}^{\infty} \Omega^k$$

$$U_j^k = \sum_{\substack{w \in \Omega^k \\ \Sigma_k(w) = j}} \text{``} D_1(w) \cdot D_2(w) \cdots D_k(w) \text{''}$$

$$D_l(w) = \begin{cases} R_{x_l} \\ \mathcal{L}_{x_l} \end{cases}$$

$$\Rightarrow \quad |D_1(w) \cdot D_2(w) \cdots D_k(w)| \le O(1) \cdot \prod |K_n - K_{n+1}|$$

$\Sigma = n$: change direction.

$$\sum_{k=0}^{\infty} U_j^k = \sum_{k=0}^{\infty} \sum_{\substack{w \in \Omega^k \\ \Sigma_k(w) = j}} \text{“} D_1(w) \cdot D_2(w) \cdots D_k(w) \text{”}$$

$\Omega^k = \bigcup_{l=0}^{\infty} \Omega_l^k$, $\quad \Omega_l^k : \{ w \in \Omega^k ;$ w change direction l times exactly $\}$.

$$\Rightarrow \sum_{k=0}^{\infty} U_j^k = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\substack{w \in \Omega_l^k \\ \Sigma_k(w) = j}} \text{“} D_1(w) \cdot D_2(w) \cdots D_k(w) \text{”}$$

$$\leq \sum_{l=0}^{\infty} \cdot \sum_{k=0}^{\infty} \cdot \sum_{\substack{w \in \Omega_l^k \\ \Sigma_k(w) = j}} O(1) \prod_m |k_n - k_{n+1}|$$

m: where $\Sigma = m$ change direction.

$$\sum_{i,j} a_i a_j = (\sum a_i)^2$$

$$\sum_{i,j,k} a_i a_j a_k = (\sum a_i)^3$$

$$\vdots \quad \vdots$$

$$= O(1) \cdot \sum_{l=0}^{\infty} \left(\sum_n |k_n - k_{n+1}| \right)^l$$ Absolutely converge.

$$\Rightarrow |\partial_x u(x,t)| \leq O(1) \frac{e^{-\frac{x^2}{C_* t}}}{t}$$ for some $C_* > 0$.

We prove when k is step function. What if $k(x)$ is B.V. funct.?

$$\begin{cases} u_t - \partial_x \big(\overset{k_n(x)}{k(x)} u_x \big) = 0 \\ u(x,0) = \delta(x - x_0) \end{cases}$$

$K(x)$: A B.V. function.

$\{k_n(x)\}_{n\in\mathbb{N}}$: A step function. A Cauchy sequence in B.V.-norm.

$$\lim_{n\to\infty} \| k_n - k \|_\infty = 0$$

Consider

$$\begin{cases} u^n_t - \partial_x (k_n(x)\, u^n_x) = 0 \\ u^n(x,0) = \delta(x - x_0) \end{cases} \Longleftarrow \overset{\text{Sol.}}{K(x,t;\, x_0, k_n) \equiv u^n(x,t)}$$

Rewrite it as:

$$u^n_\tau - \partial_y (k_n(y) \cdot u^n_y) = 0$$

$$\Rightarrow \int_0^t \int_{\mathbb{R}} K(x, t-\tau;\, y;\, k_n) \cdot (u^n_\tau - \partial_y (k_n(y) \cdot u^n_y))\, dy\, d\tau = 0$$

$$u^n(x,t) - \int_{\mathbb{R}} K(x,t;\, y, k_n) \cdot u(y,0)\, dy$$

$$+ \int_0^t \int_{\mathbb{R}} - \partial_\tau K(x, t-\tau;\, y, k_n)\, u^n$$

$$+ K_y(x, t-\tau;\, y, k_n) \cdot \underline{k_n(y)} \cdot u^n_y \, dy\, d\tau = 0$$

$(k_m$

Question 1:

Consider the problem

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0 \quad \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{cases} \qquad (1)$$

Use the singular-regular decomposition to decompose the solution $u(x,t) = u_*(x,t) + u_\#(x,t)$, where $u_*(x,t)$ is a singular part with an exponentaily decaying structure in both x and t variable; and $u_\#(x,t)$ is a regular part with a sufficient regularity in x variable.

(1) Use energy estimate to show that $u_\#(x,t)$ is exponentially decaying in x variable when $|x| > 2t$ with $t > 1$.

(2) Use Long wave-short wave decomposition, Fourier transform, energy estimates, and complex analysis to show that there exists $C > 0$ s.t.

$$|u_\#(x,t)| < C \cdot \left(\frac{e^{-\frac{x^2}{C(t+1)}}}{\sqrt{t+1}} \right) \qquad (2).$$

Question 2:

Let $\mathcal{L}_t[u](x,s)$ be the Laplace transform of $u(x,t)$ w.r.t. t

$\mathcal{L}_t[u](x,s) = \int_0^\infty u(x,t) \cdot e^{-st}\, dt$ with $\mathrm{Re}(s) \geq 0$. Let $u(x,t)$ be the

Solution of (1) and compute $\mathcal{L}_t[u](x,s)$ in terms of Laplace wave

trains:

$$\mathcal{L}_t[u](x,s) = \begin{cases} A_+(s) \cdot e^{\lambda_+(s) x} & \text{for } x > 0 \\ A_-(s) \cdot e^{\lambda_-(s) x} & \text{for } x < 0 \end{cases}$$

(1) Find $\lambda_\pm(s)$ and $A_\pm(s)$. Let $w(x,t)$ be the solution of the initial boundary value problem:

$$\begin{cases} w_{tt} - w_{xx} + w_t = 0 \quad \text{for } x, t > 0, \\ w(x,0) = w(0,t) = 0, \\ w_t(x,0) = \delta(x - x_0) \quad \text{with } x_0 > 0 \end{cases}$$

(2) Find the solution $\mathcal{L}[w](x,s)$ in terms of the Laplace wave trains $e^{\lambda_+(s)(x-x_0)}$, $e^{\lambda_-(s)(x-x_0)}$ and $e^{\lambda_+(s) x}$.

Question 3:

Let $u(x,t)$ be the weak solution of the heat equation

$$\begin{cases} \partial_t u - \partial_x \mu(x) \partial_x u = 0 \\ u(x,0) = \delta(x) \end{cases}$$

where $\mu(x) \equiv 1 + H(x+1) - H(x-1)$, and $H(x)$ is the Heaveside funct.

i.e. $H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

- Show that $\exists\, C > 0$ s.t.

$$|u_x(x,t)| \leq C \cdot \frac{e^{-\frac{x^2}{Ct}}}{t}.$$