A Hamilton-Jacobi Approach for Asymptotic Propagation Shape of a Road-field Model

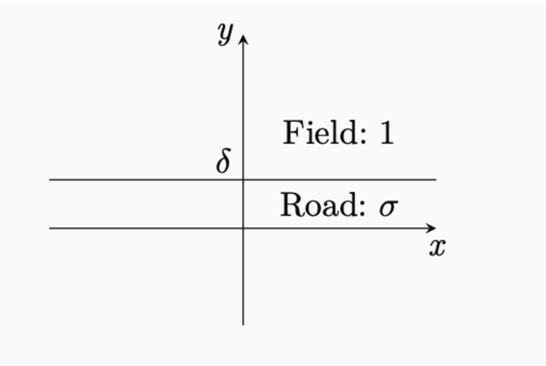
Xingri Geng

National University of Singapore

Nov 04, 2022

Overview

Motivations : a road-field model



- Population growth with a road.

Scenario :

- Road is narrow, and field is large ;
- diffusion rate is large on road, small in field.
- Multi-scales in both spatial variable and diffusion rate.
- Thus cumbersome and difficult to solve the "full model" ;
- hard to see the effects of the road.

Resolution : Think of the road as a widthless line and then impose suitable conditions on it.

Scenario :

- Road is narrow, and field is large ;
- diffusion rate is large on road, small in field.
- Multi-scales in both spatial variable and diffusion rate.
- Thus cumbersome and difficult to solve the "full model" ;
- hard to see the effects of the road.

Resolution : Think of the road as a widthless line and then impose suitable conditions on it.

Scenario :

- Road is narrow, and field is large ;
- diffusion rate is large on road, small in field.
- Multi-scales in both spatial variable and diffusion rate.
- Thus cumbersome and difficult to solve the "full model" ;
- hard to see the effects of the road.

Resolution : Think of the road as a widthless line and then impose suitable conditions on it.

Scenario :

- Road is narrow, and field is large ;
- diffusion rate is large on road, small in field.
- Multi-scales in both spatial variable and diffusion rate.
- Thus cumbersome and difficult to solve the “full model” ;
- hard to see the effects of the road.

Resolution : Think of the road as a widthless line and then impose suitable conditions on it.

Scenario :

- Road is narrow, and field is large ;
- diffusion rate is large on road, small in field.
- Multi-scales in both spatial variable and diffusion rate.
- Thus cumbersome and difficult to solve the “full model” ;
- hard to see the effects of the road.

Resolution : Think of the road as a widthless line and then impose suitable conditions on it.

Scenario :

- Road is narrow, and field is large ;
- diffusion rate is large on road, small in field.
- Multi-scales in both spatial variable and diffusion rate.
- Thus cumbersome and difficult to solve the "full model" ;
- hard to see the effects of the road.

Resolution : Think of the road as a widthless line and then impose suitable conditions on it.

Motivation

In 2013, Berestycki, Roquejoffre and Rossi proposed a simple model :

- the road has no width — a line ;
- no reproduction on the line ;
- exchange between the line and the field ;
- symmetry w.r.t. the y direction.

Motivation

In 2013, Berestycki, Roquejoffre and Rossi proposed a simple model :

- the road has no width — a line ;
- no reproduction on the line ;
- exchange between the line and the field ;
- symmetry w.r.t. the y direction.

Motivation

In 2013, Berestycki, Roquejoffre and Rossi proposed a simple model :

- the road has no width — a line ;
- no reproduction on the line ;
- exchange between the line and the field ;
- symmetry w.r.t. the y direction.

Motivation

In 2013, Berestycki, Roquejoffre and Rossi proposed a simple model :

- the road has no width — a line ;
- no reproduction on the line ;
- exchange between the line and the field ;
- symmetry w.r.t. the y direction.

Motivation

In 2013, Berestycki, Roquejoffre and Rossi proposed a simple model :

- the road has no width — a line ;
- no reproduction on the line ;
- exchange between the line and the field ;
- symmetry w.r.t. the y direction.

Motivation

Berestycki, Roquejoffre and Rossi [1] proposed the following system

$$\begin{cases} u_t - Du_{xx} = \nu v(x,0,t) - \mu u, & x \in \mathbb{R}, t > 0, \\ v_t - \Delta v = v(1-v), & (x,y) \in \mathbb{R}^2_+, t > 0, \\ -dv_y(x,0,t) = \mu u - \nu v(x,0,t), & x \in \mathbb{R}, t > 0. \end{cases}$$

u : line density of species on the line.
v : area density of species in the field.

- The asymptotic speed of spreading $c_* = c_{KPP} = 2$ if $D \leq 2$; $c_* > 2$ if $D > 2$ and $c_* = O(\sqrt{D})$ as $D \to \infty$;
- and

$$\lim_{t\to\infty} \sup_{|x|\geq ct} (u(x,y,t), v(x,y,t)) = (0,0) \text{ for any } c > c_*,$$

$$\lim_{t\to\infty} \sup_{|x|\leq ct} (u(x,y,t), v(x,y,t)) = (1/\mu, 1) \text{ for any } 0 < c < c_*.$$

1. J. Math. Biol., 2013

Motivation

Berestycki, Roquejoffre and Rossi [1] proposed the following system

$$\begin{cases} u_t - Du_{xx} = \nu v(x,0,t) - \mu u, & x \in \mathbb{R}, t > 0, \\ v_t - \Delta v = v(1-v), & (x,y) \in \mathbb{R}^2_+, t > 0, \\ -dv_y(x,0,t) = \mu u - \nu v(x,0,t), & x \in \mathbb{R}, t > 0. \end{cases}$$

u : line density of species on the line.
v : area density of species in the field.

- The asymptotic speed of spreading $c_* = c_{KPP} = 2$ if $D \leq 2$; $c_* > 2$ if $D > 2$ and $c_* = O(\sqrt{D})$ as $D \to \infty$;
- and

$$\lim_{t\to\infty} \sup_{|x|\geq ct} (u(x,y,t), v(x,y,t)) = (0,0) \text{ for any } c > c_*,$$

$$\lim_{t\to\infty} \sup_{|x|\leq ct} (u(x,y,t), v(x,y,t)) = (1/\mu, 1) \text{ for any } 0 < c < c_*.$$

1. J. Math. Biol., 2013

Motivation

Berestycki, Roquejoffre and Rossi [1] proposed the following system

$$\begin{cases} u_t - Du_{xx} = \nu v(x,0,t) - \mu u, & x \in \mathbb{R}, t > 0, \\ v_t - \Delta v = v(1-v), & (x,y) \in \mathbb{R}^2_+, t > 0, \\ -dv_y(x,0,t) = \mu u - \nu v(x,0,t), & x \in \mathbb{R}, t > 0. \end{cases}$$

u : line density of species on the line.
v : area density of species in the field.

- The asymptotic speed of spreading $c_* = c_{KPP} = 2$ if $D \leq 2$; $c_* > 2$ if $D > 2$ and $c_* = O(\sqrt{D})$ as $D \to \infty$;
- and

$$\lim_{t\to\infty} \sup_{|x|\geq ct} (u(x,y,t), v(x,y,t)) = (0,0) \text{ for any } c > c_*,$$

$$\lim_{t\to\infty} \sup_{|x|\leq ct} (u(x,y,t), v(x,y,t)) = (1/\mu, 1) \text{ for any } 0 < c < c_*.$$

1. J. Math. Biol., 2013

A different model

A different model by using the idea of "effective boundary conditions"[2].

- Start with the full model ;
- send $\delta \to 0$, and prove the solution of full model converges, then get the "effective model"(limiting model) with "effective boundary condition"(EBC) imposed on $x-$axis.
- The multiple scales are all gone in the effective model .

2. H. Li and X. Wang, Nonlinearity, 2017

A different model

A different model by using the idea of "effective boundary conditions"[2].

- Start with the full model ;
- send $\delta \to 0$, and prove the solution of full model converges, then get the "effective model"(limiting model) with "effective boundary condition"(EBC) imposed on $x-$axis.
- The multiple scales are all gone in the effective model .

2. H. Li and X. Wang, Nonlinearity, 2017

A different model

A different model by using the idea of "effective boundary conditions"[2].

- Start with the full model ;
- send $\delta \to 0$, and prove the solution of full model converges, then get the "effective model"(limiting model) with "effective boundary condition"(EBC) imposed on $x-$axis.
- The multiple scales are all gone in the effective model .

2. H. Li and X. Wang, Nonlinearity, 2017

A different model

The full model :

$$\begin{cases} u_t - \nabla \cdot (\tilde{\sigma} \nabla u) = u(1-u), & (x,y) \in \mathbb{R}^2, t > 0, \\ u(x,y,0) = \varphi(x,y), & (x,y) \in \mathbb{R}^2, \\ 0 \leq \varphi \leq 1, \;\; \varphi \not\equiv 0, \;\; \varphi \text{ compactly supported} \end{cases}$$

where

$$\tilde{\sigma} = \begin{cases} \sigma, & \text{if } y \in (0, \delta), \\ 1, & \text{else.} \end{cases}$$

The cases of $\sigma \geq O(1)$ and $\sigma = o(1)$ are studied.

- *Remark : The Berestycki-Roquejoffre-Rossi model can also be derived by using the idea of effective boundary conditions from a different full model.*

A different model

The full model :

$$\begin{cases} u_t - \nabla \cdot (\tilde{\sigma} \nabla u) = u(1-u), & (x,y) \in \mathbb{R}^2, t > 0, \\ u(x,y,0) = \varphi(x,y), & (x,y) \in \mathbb{R}^2, \\ 0 \leq \varphi \leq 1, \ \ \varphi \not\equiv 0, \ \ \varphi \text{ compactly supported} & \end{cases}$$

where

$$\tilde{\sigma} = \begin{cases} \sigma, & \text{if } y \in (0, \delta), \\ 1, & \text{else.} \end{cases}$$

The cases of $\sigma \geq O(1)$ and $\sigma = o(1)$ are studied.

- *Remark : The Berestycki-Roquejoffre-Rossi model can also be derived by using the idea of effective boundary conditions from a different full model.*

A different model

The full model :

$$\begin{cases} u_t - \nabla \cdot (\tilde{\sigma} \nabla u) = u(1-u), & (x,y) \in \mathbb{R}^2, t > 0, \\ u(x,y,0) = \varphi(x,y), & (x,y) \in \mathbb{R}^2, \\ 0 \leq \varphi \leq 1, \ \ \varphi \not\equiv 0, \ \ \varphi \text{ compactly supported} \end{cases}$$

where

$$\tilde{\sigma} = \begin{cases} \sigma, & \text{if } y \in (0,\delta), \\ 1, & \text{else.} \end{cases}$$

The cases of $\sigma \geq O(1)$ and $\sigma = o(1)$ are studied.

- *Remark : The Berestycki-Roquejoffre-Rossi model can also be derived by using the idea of effective boundary conditions from a different full model.*

Main Result [3]

Theorem

$\forall T \in (0,\infty)$, $u \to v$ in $C\left([0,T], L^2_{\text{loc}}(\mathbb{R}^2)\right)$ as $\delta \to 0$, where v satisfies

$$v_t = \Delta v + v(1-v), \; y \neq 0, \;\; t > 0,$$

the same initial condition but the following EBC on x-axis.

3. H. Li and X. Wang, Nonlinearity, 2017

EBC with Large σ

As $\delta \to 0$	EBC on $y = 0$
$\sigma \geq O(1) > 0$, $\sigma\delta \to 0$	$v^+ = v^-$, $v_y^+ = v_y^-$
$\sigma\delta \to a \in (0, \infty)$	$v^+ = v^-, v_y^- - v_y^+ = av_{xx}^+$
$\sigma\delta \to \infty$	$v^+ = v^- = 0$

EBC with Small σ

As $\delta \to 0$	EBC on $y = 0$
$\frac{\sigma}{\delta} \to 0$	$v_y^- = v_y^+ = 0$
$\frac{\sigma}{\delta} \to b \in (0, \infty)$	$v_y^+ = v_y^-$, $v_y^- = b(v^+ - v^-)$
$\frac{\sigma}{\delta} \to \infty$, $\sigma \to 0$	$v^+ = v^-$, $v_y^+ = v_y^-$

Asymptotic propagation speed

- A natural question : what is the asymptotic propagation speed and shape of the effective model with the boundary condition

$$v^+ = v^-, v_y^- - v_y^+ = av_{xx}^+ \quad ?$$

- Answered by X.F. Chen, J.F. He, and X.F. Wang[4].
- A Hamilton-Jacobi approach for asymptotic propagation speed and shape.

4. The asymptotic propagation speed of the Fisher-KPP equation with effective boundary condition on a road, preprint

Asymptotic propagation speed

- A natural question : what is the asymptotic propagation speed and shape of the effective model with the boundary condition

$$v^+ = v^-, v_y^- - v_y^+ = a v_{xx}^+ \quad ?$$

- Answered by X.F. Chen, J.F. He, and X.F. Wang[4].

- A Hamilton-Jacobi approach for asymptotic propagation speed and shape.

4. The asymptotic propagation speed of the Fisher-KPP equation with effective boundary condition on a road, preprint

Asymptotic propagation speed

- A natural question : what is the asymptotic propagation speed and shape of the effective model with the boundary condition

$$v^+ = v^-, v_y^- - v_y^+ = av_{xx}^+ \quad ?$$

- Answered by X.F. Chen, J.F. He, and X.F. Wang [4].

- A Hamilton-Jacobi approach for asymptotic propagation speed and shape.

4. The asymptotic propagation speed of the Fisher-KPP equation with effective boundary condition on a road, preprint

History

There are lots of works on propagation for the reaction-diffusion PDE using a Hamilton-Jacob approach.

- In 1989, L.C. Evans and P. E. Souganidis studied certain nonlinear reaction-diffusion equations.
- In 1990, G. Barles, L.C. Evans and P. E. Souganidis studied wavefront propagation for reaction-diffusion systems.
- In 2020, Q. Liu, S. Liu, and K.-Y. Lam studied a two-species Lotka-Volterra competition-diffusion system.
- In 2021, Q. Liu, S. Liu, and K.-Y. Lam studied a three-species competition system.
- In 2022, K.-Y. Lam, Y. Lou and B. Perthame studied evolution of dispersal.
- ...

History

There are lots of works on propagation for the reaction-diffusion PDE using a Hamilton-Jacob approach.

- In 1989, L.C. Evans and P. E. Souganidis studied certain nonlinear reaction-diffusion equations.
- In 1990, G. Barles, L.C. Evans and P. E. Souganidis studied wavefront propagation for reaction-diffusion systems.
- In 2020, Q. Liu, S. Liu, and K.-Y. Lam studied a two-species Lotka-Volterra competition-diffusion system.
- In 2021, Q. Liu, S. Liu, and K.-Y. Lam studied a three-species competition system.
- In 2022, K.-Y. Lam, Y. Lou and B. Perthame studied evolution of dispersal.
- ...

History

There are lots of works on propagation for the reaction-diffusion PDE using a Hamilton-Jacob approach.

- In 1989, L.C. Evans and P. E. Souganidis studied certain nonlinear reaction-diffusion equations.
- In 1990, G. Barles, L.C. Evans and P. E. Souganidis studied wavefront propagation for reaction-diffusion systems.
- In 2020, Q. Liu, S. Liu, and K.-Y. Lam studied a two-species Lotka-Volterra competition-diffusion system.
- In 2021, Q. Liu, S. Liu, and K.-Y. Lam studied a three-species competition system.
- In 2022, K.-Y. Lam, Y. Lou and B. Perthame studied evolution of dispersal.
- ...

History

There are lots of works on propagation for the reaction-diffusion PDE using a Hamilton-Jacob approach.

- In 1989, L.C. Evans and P. E. Souganidis studied certain nonlinear reaction-diffusion equations.
- In 1990, G. Barles, L.C. Evans and P. E. Souganidis studied wavefront propagation for reaction-diffusion systems.
- In 2020, Q. Liu, S. Liu, and K.-Y. Lam studied a two-species Lotka-Volterra competition-diffusion system.
- In 2021, Q. Liu, S. Liu, and K.-Y. Lam studied a three-species competition system.
- In 2022, K.-Y. Lam, Y. Lou and B. Perthame studied evolution of dispersal.
- ...

History

There are lots of works on propagation for the reaction-diffusion PDE using a Hamilton-Jacob approach.

- In 1989, L.C. Evans and P. E. Souganidis studied certain nonlinear reaction-diffusion equations.
- In 1990, G. Barles, L.C. Evans and P. E. Souganidis studied wavefront propagation for reaction-diffusion systems.
- In 2020, Q. Liu, S. Liu, and K.-Y. Lam studied a two-species Lotka-Volterra competition-diffusion system.
- In 2021, Q. Liu, S. Liu, and K.-Y. Lam studied a three-species competition system.
- In 2022, K.-Y. Lam, Y. Lou and B. Perthame studied evolution of dispersal.
- ...

History

There are lots of works on propagation for the reaction-diffusion PDE using a Hamilton-Jacob approach.

- In 1989, L.C. Evans and P. E. Souganidis studied certain nonlinear reaction-diffusion equations.
- In 1990, G. Barles, L.C. Evans and P. E. Souganidis studied wavefront propagation for reaction-diffusion systems.
- In 2020, Q. Liu, S. Liu, and K.-Y. Lam studied a two-species Lotka-Volterra competition-diffusion system.
- In 2021, Q. Liu, S. Liu, and K.-Y. Lam studied a three-species competition system.
- In 2022, K.-Y. Lam, Y. Lou and B. Perthame studied evolution of dispersal.
- ...

Effective Problem

The effective problem :

$$\begin{cases} u_t - \Delta u = u(1-u), & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![u]\!] = 0, [\![u_y]\!] = -2au_{xx}, & x \in \mathbb{R}, y = 0, t > 0, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where

$$[\![u]\!] := u(x,0+,t) - u(x,0-,t)$$

and $0 \leq \varphi \leq 1$, $\varphi \not\equiv 0$, φ compactly supported.

- Question : what is the asymptotic propagation speed and shape of this model ?

Effective Problem

The effective problem :

$$\begin{cases} u_t - \Delta u = u(1-u), & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![u]\!] = 0, [\![u_y]\!] = -2a u_{xx}, & x \in \mathbb{R}, y = 0, t > 0, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where

$$[\![u]\!] := u(x, 0+, t) - u(x, 0-, t)$$

and $0 \leq \varphi \leq 1$, $\varphi \not\equiv 0$, φ compactly supported.

- Question : what is the asymptotic propagation speed and shape of this model ?

Asymptotic speed and shape

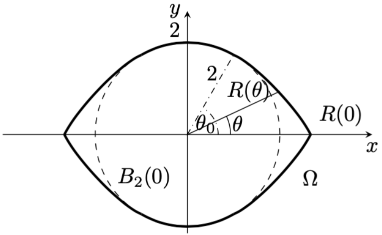
Theorem

For each $\nu \in (0,1)$,

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega^c(t))} = 0, \quad \lim_{t\to\infty} \|u(\cdot,t) - 1\|_{L^\infty(\Omega(\nu t))} = 0,$$

where $\Omega(t) = t\Omega(1)$ *and* $\Omega(1) = \{(x,y)|\varphi^*(x,y,1) < 1\}$,

$$\varphi^*(x,y,t) := \min_{s\geq 0}\left\{\frac{x^2}{4(t+as)} + \frac{(|y|+s)^2}{4t}\right\}.$$



$$\theta_0 = arcsin\frac{2a}{1+\sqrt{1+4a^2}}$$

Effective Problem

Inspired by L. C. Evans [5] and Souganidis [6], consider the following rescaling

$$u^\varepsilon(x,y,t) = u\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon}\right),$$

where $u^\varepsilon(x,y,t)$ satisfies

$$\begin{cases} u_t^\varepsilon = \varepsilon \Delta u^\varepsilon + \dfrac{1}{\varepsilon} u^\varepsilon (1-u^\varepsilon), & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![u^\varepsilon]\!] = 0, [\![u_y^\varepsilon]\!] = -2a\varepsilon u_{xx}^\varepsilon, & x \in \mathbb{R}, y = 0, t > 0, \\ u^\varepsilon(x,y,0) = u_0^\varepsilon(x/\varepsilon, y/\varepsilon) := g^\varepsilon(x,y), & (x,y) \in \mathbb{R}^2. \end{cases} \tag{1.2}$$

5. Indiana Univ. Math, 1989
6. Springer, Berlin, Heidelberg, 1997

Effective Problem

Lemma (1)

There exists a constant C independent of $\varepsilon > 0$, such that

$$0 < u^\varepsilon \leq C.$$

and

$$\lim_{\varepsilon \to 0} sup \quad u^\varepsilon \leq 1$$

locally uniformly in $\mathbb{R}^2 \times \mathbb{R}_+$.

The phase function

- Perform the logarithmic transformation[7] : $v^\varepsilon = -\varepsilon \log u^\varepsilon$.
- v^ε satisfies the following Hamilton-Jacobi equation :

$$\begin{cases} v_t^\varepsilon - \varepsilon\Delta v^\varepsilon + |\nabla v^\varepsilon|^2 + 1 - e^{-v^\varepsilon/\varepsilon} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![v^\varepsilon]\!] = 0, [\![v_y^\varepsilon]\!] = -2a\left[\varepsilon v_{xx}^\varepsilon - (v_x^\varepsilon)^2\right], & x \in \mathbb{R}, y = 0, t > 0, \\ v^\varepsilon(x,y,0) = -\varepsilon \log g^\varepsilon, & (x,y) \in G_\varepsilon := spt\{g^\varepsilon\}, \\ v^\varepsilon \to \infty \text{ as } t \to 0+, & (x,y) \in \mathbb{R}^2 \backslash G_\varepsilon, \end{cases} \tag{1.3}$$

- v^ε converges to some v locally uniformly (to be proved later), where v satisfies

$$\begin{cases} \min\{v_t + |\nabla v|^2 + 1, v\} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![v]\!] = 0, [\![v_y]\!] = 2av_x^2, & x \in \mathbb{R}, y = 0, t > 0, \\ v(x,y,0) = 0, & (x,y) = (0,0), \\ v \to \infty \text{ as } t \to 0+, & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \end{cases} \tag{HJ}$$

7. M. I. Freidlin, Ann. Probab., 1985

The phase function

- Perform the logarithmic transformation [7] : $v^\varepsilon = -\varepsilon \log u^\varepsilon$.
- v^ε satisfies the following Hamilton-Jacobi equation :

$$\begin{cases} v_t^\varepsilon - \varepsilon\Delta v^\varepsilon + |\nabla v^\varepsilon|^2 + 1 - e^{-v^\varepsilon/\varepsilon} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![v^\varepsilon]\!] = 0, [\![v_y^\varepsilon]\!] = -2a\left[\varepsilon v_{xx}^\varepsilon - (v_x^\varepsilon)^2\right], & x \in \mathbb{R}, y = 0, t > 0, \\ v^\varepsilon(x,y,0) = -\varepsilon \log g^\varepsilon, & (x,y) \in G_\varepsilon := spt\{g^\varepsilon\}, \\ v^\varepsilon \to \infty \text{ as } t \to 0+, & (x,y) \in \mathbb{R}^2 \backslash G_\varepsilon, \end{cases} \tag{1.3}$$

- v^ε converges to some v locally uniformly (to be proved later), where v satisfies

$$\begin{cases} \min\{v_t + |\nabla v|^2 + 1, v\} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![v]\!] = 0, [\![v_y]\!] = 2av_x^2, & x \in \mathbb{R}, y = 0, t > 0, \\ v(x,y,0) = 0, & (x,y) = (0,0), \\ v \to \infty \text{ as } t \to 0+, & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \end{cases} \tag{HJ}$$

7. M. I. Freidlin, Ann. Probab., 1985

The phase function

- Perform the logarithmic transformation [7] : $v^\varepsilon = -\varepsilon \log u^\varepsilon$.
- v^ε satisfies the following Hamilton-Jacobi equation :

$$\begin{cases} v_t^\varepsilon - \varepsilon\Delta v^\varepsilon + |\nabla v^\varepsilon|^2 + 1 - e^{-v^\varepsilon/\varepsilon} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![v^\varepsilon]\!] = 0, [\![v_y^\varepsilon]\!] = -2a\left[\varepsilon v_{xx}^\varepsilon - (v_x^\varepsilon)^2\right], & x \in \mathbb{R}, y = 0, t > 0, \\ v^\varepsilon(x,y,0) = -\varepsilon \log g^\varepsilon, & (x,y) \in G_\varepsilon := spt\{g^\varepsilon\}, \\ v^\varepsilon \to \infty \text{ as } t \to 0+, & (x,y) \in \mathbb{R}^2 \backslash G_\varepsilon, \end{cases} \tag{1.3}$$

- v^ε converges to some v locally uniformly (to be proved later), where v satisfies

$$\begin{cases} \min\{v_t + |\nabla v|^2 + 1, v\} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ [\![v]\!] = 0, [\![v_y]\!] = 2av_x^2, & x \in \mathbb{R}, y = 0, t > 0, \\ v(x,y,0) = 0, & (x,y) = (0,0), \\ v \to \infty \text{ as } t \to 0+, & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \end{cases} \tag{HJ}$$

7. M. I. Freidlin, Ann. Probab., 1985

Lemma (2)

For any compact subset $K \in \mathbb{R}^2 \times \mathbb{R}_+$, *there exists a constant* $C(K)$ *independent of* $\varepsilon > 0$, *such that*

$$\sup_K |v^\varepsilon| \leq C(K).$$

Define the so-called half-relaxed limits

- $v^*(x,t) = \limsup_{\substack{\varepsilon \to 0 \\ (x',t') \to (x,t)}} v^\varepsilon(x',t')$
- $v_*(x,t) = \liminf_{\substack{\varepsilon \to 0 \\ (x',t') \to (x,t)}} v^\varepsilon(x',t')$

Lemma (2)

For any compact subset $K \in \mathbb{R}^2 \times \mathbb{R}_+$, there exists a constant $C(K)$ independent of $\varepsilon > 0$, such that

$$\sup_K |v^\varepsilon| \leq C(K).$$

Define the so-called half-relaxed limits

- $v^*(x,t) = \limsup_{\substack{\varepsilon \to 0 \\ (x',t') \to (x,t)}} v^\varepsilon(x',t')$
- $v_*(x,t) = \liminf_{\substack{\varepsilon \to 0 \\ (x',t') \to (x,t)}} v^\varepsilon(x',t')$

Variational Inequality

- Consider the following variational inequality :

$$\begin{cases} \min\{\mathcal{T}[v], v\} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ \mathcal{B}[v] = 0, & x \in \mathbb{R}, y = 0, t > 0, \\ v(x, y, 0) = g_0(x, y), & x \in \mathbb{R}^2, \end{cases} \tag{2.1}$$

where $\mathcal{T}[v] = v_t + |\nabla v|^2 + 1$, $\mathcal{B}[v] = 2av_x^2 - [\![v_y]\!]$, and g_0 is positive, bounded, and Liptschitz continuous on $\mathbb{R}^2$.

- Use the theory of viscosity solutions [8].

8. M. G Crandall, H. Ishii, and P-L. Lions., Bull. Amer. Math. Soc., 1992

Variational Inequality

- Consider the following variational inequality :

$$\begin{cases} \min\{\mathcal{T}[v], v\} = 0, & x \in \mathbb{R}, y \neq 0, t > 0, \\ \mathcal{B}[v] = 0, & x \in \mathbb{R}, y = 0, t > 0, \\ v(x, y, 0) = g_0(x, y), & x \in \mathbb{R}^2, \end{cases} \tag{2.1}$$

where $\mathcal{T}[v] = v_t + |\nabla v|^2 + 1$, $\mathcal{B}[v] = 2av_x^2 - [\![v_y]\!]$, and g_0 is positive, bounded, and Liptschitz continuous on $\mathbb{R}^2$.

- Use the theory of viscosity solutions [8].

8. M. G Crandall, H. Ishii, and P-L. Lions., Bull. Amer. Math. Soc., 1992

Viscosity solutions

Definition

A upper (*lower*) semi-continuous function $\underline{v}$ ($\overline{v}$) is a viscosity subsolution (*supersolution*) of (2.1) on $\mathbb{R}^2 \times \mathbb{R}_+$ if for any $\phi \in C^1(\overline{\mathbb{R}^2_+} \times \mathbb{R}_+) \cap C^1(\overline{\mathbb{R}^2_-} \times \mathbb{R}_+) \cap C(\overline{\mathbb{R}^2} \times \mathbb{R}_+)$, assume $\underline{v} - \phi$ ($\overline{v} - \phi$) attains a strict maximum (*minimum*) at some $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+$ and $\underline{v} > 0(\overline{v} > 0)$, then we have $\mathcal{T}[\phi](x, y, t) \leq 0(\geq 0)$ if $y \neq 0$, and if $y = 0$,

$$\min\{\mathcal{T}[\phi](x, 0\pm, t), \mathcal{B}[\phi](x, 0, t)\} \leq 0(\geq 0).$$

- v is a viscosity solution of (2.1) if v is both a subsolution and supersolution.

Viscosity solutions

Definition

A upper (*lower*) semi-continuous function $\underline{v}$ ($\overline{v}$) is a viscosity subsolution (*supersolution*) of (2.1) on $\mathbb{R}^2 \times \mathbb{R}_+$ if for any $\phi \in C^1(\overline{\mathbb{R}^2_+} \times \mathbb{R}_+) \cap C^1(\overline{\mathbb{R}^2_-} \times \mathbb{R}_+) \cap C(\overline{\mathbb{R}^2} \times \mathbb{R}_+)$, assume $\underline{v} - \phi$ ($\overline{v} - \phi$) attains a strict maximum (*minimum*) at some $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+$ and $\underline{v} > 0 (\overline{v} > 0)$, then we have $\mathcal{T}[\phi](x, y, t) \leq 0 (\geq 0)$ if $y \neq 0$, and if $y = 0$,

$$\min\{\mathcal{T}[\phi](x, 0\pm, t), \mathcal{B}[\phi](x, 0, t)\} \leq 0 (\geq 0).$$

- v is a viscosity solution of (2.1) if v is both a subsolution and supersolution.

Comparison principle

Theorem (2.1)

Let $\underline{v}$ and $\overline{v}$ be a viscosity sub solution and super solution of (2.1) respectively. If $\underline{v}$ is bounded above and $\overline{v}$ is bounded below, and $\underline{v}(x, y, 0) \leq \overline{v}(x, y, 0)$, then $\underline{v} \leq \overline{v}$ on $\mathbb{R}^2 \times \overline{\mathbb{R}_+}$.

Outline of the proof

- Step 1. Obtain the estimate $\sup_K |v^\varepsilon| \leq C(K)$;
- Step 2. Show that v^* and v_* are subsolution and supersolution of (HJ), respectively ; moreover, $v_* \geq v^*$.
- Step 3. v^ε converges to v uniformly locally, and hence v is a viscosity solution of (1.3).

Outline of the proof

- Step 1. Obtain the estimate $\sup_K |v^\varepsilon| \leq C(K)$;
- Step 2. Show that v^* and v_* are subsolution and supersolution of (HJ), respectively ; moreover, $v_* \geq v^*$.
- Step 3. v^ε converges to v uniformly locally, and hence v is a viscosity solution of (1.3).

Outline of the proof

- Step 1. Obtain the estimate $\sup_K |v^\varepsilon| \leq C(K)$;
- Step 2. Show that v^* and v_* are subsolution and supersolution of (HJ), respectively; moreover, $v_* \geq v^*$.
- Step 3. v^ε converges to v uniformly locally, and hence v is a viscosity solution of (1.3).

An optimal control problem

Recall the variational inequality (2.1), the viscosity solution exists and is unique[9].

- For any $p = (p_1, p_2) \in \mathbb{R}^2$, define

$$H(p) = |p|^2 + 1, \quad B(p_1) = ap_1^2.$$

- Legendre transformation in the field and road are :

$$\begin{aligned} L(q) &= \sup_{p \in \mathbb{R}^2} \{q \cdot p - H(p)\} = \frac{|q|^2}{4} - 1, \\ G(q_1) &= \sup_{p_1 \in \mathbb{R}} \{q_1 \cdot p_1 - B(p_1)\} = \frac{q_1^2}{4a}. \end{aligned} \tag{3.1}$$

9. Indiana Univ. Math, 1989.

An optimal control problem

Introduce a two player, zero-sum differential game :

- consider an ordinary differential equation

$$\begin{cases} \dot{\gamma}(\tau) = f(\tau, \gamma(\tau), \eta(\tau), l(\tau)), & \tau \in (0, t), \\ \gamma(0) = x, & x \in \mathbb{R} \times \{y \neq 0\}. \end{cases} \tag{3.2}$$

- *Player 1 : Find controls to minimize running-cost :*

$$\int_0^t L(-\eta(\tau)) + F(\gamma, \eta, l)(\tau) d\tau,$$

where $F(\gamma, \eta, l)(\tau)$ is the running-cost on the x-axis.

- *Player 2 : Stop the game and maximize running-cost.*

An optimal control problem

Introduce a two player, zero-sum differential game :

- consider an ordinary differential equation

$$\begin{cases} \dot{\gamma}(\tau) = f(\tau, \gamma(\tau), \eta(\tau), l(\tau)), & \tau \in (0,t), \\ \gamma(0) = x, & x \in \mathbb{R} \times \{y \neq 0\}. \end{cases} \tag{3.2}$$

- *Player 1 : Find controls to minimize running-cost :*

$$\int_0^t L(-\eta(\tau)) + F(\gamma, \eta, l)(\tau) d\tau,$$

where $F(\gamma, \eta, l)(\tau)$ is the running-cost on the x-axis.

- *Player 2 : Stop the game and maximize running-cost.*

An optimal control problem

Introduce a two player, zero-sum differential game :

- consider an ordinary differential equation

$$\begin{cases} \dot{\gamma}(\tau) = f(\tau, \gamma(\tau), \eta(\tau), l(\tau)), & \tau \in (0, t), \\ \gamma(0) = x, & x \in \mathbb{R} \times \{y \neq 0\}. \end{cases} \tag{3.2}$$

- *Player 1 : Find controls to minimize running-cost :*

$$\int_0^t L(-\eta(\tau)) + F(\gamma, \eta, l)(\tau) d\tau,$$

where $F(\gamma, \eta, l)(\tau)$ is the running-cost on the x-axis.

- *Player 2 : Stop the game and maximize running-cost.*

Optimal Controls (*Player 1*)

More preciously, define a control triple (γ, η, l) :

- Control path $\gamma = (\gamma_1, \gamma_2) \in AC([0,t], \mathbb{R}^2)$, $\gamma(0) = x$ in the upper half plane, and $\gamma(\tau) \in \overline{\mathbb{R} \times \mathbb{R}_+}$;
- Control velocity : $\eta = (\eta_1, \eta_2) \in L^2([0,t], \mathbb{R}^2)$;
- Local time : $l \in L^2([0,t], \mathbb{R})$,

$$\begin{cases} l(\tau) \geq 0 \text{ for } a.e. \tau \in [0,t] \\ l(\tau) = 0 \text{ if } l(\tau) \in \{y > 0\}. \end{cases}$$

More preciously, define a control triple (γ, η, l) :

- Control path $\gamma = (\gamma_1, \gamma_2) \in AC([0,t], \mathbb{R}^2)$, $\gamma(0) = x$ in the upper half plane, and $\gamma(\tau) \in \overline{\mathbb{R} \times \mathbb{R}_+}$;
- Control velocity : $\eta = (\eta_1, \eta_2) \in L^2([0,t], \mathbb{R}^2)$;
- Local time : $l \in L^2([0,t], \mathbb{R})$,

$$\begin{cases} l(\tau) \geq 0 \text{ for } a.e. \tau \in [0,t] \\ l(\tau) = 0 \text{ if } l(\tau) \in \{y > 0\}. \end{cases}$$

Optimal Controls (*Player 1*)

More preciously, define a control triple (γ, η, l) :

- Control path $\gamma = (\gamma_1, \gamma_2) \in AC([0,t], \mathbb{R}^2)$, $\gamma(0) = x$ in the upper half plane, and $\gamma(\tau) \in \overline{\mathbb{R} \times \mathbb{R}_+}$;
- Control velocity : $\eta = (\eta_1, \eta_2) \in L^2([0,t], \mathbb{R}^2)$;
- Local time : $l \in L^2([0,t], \mathbb{R})$,

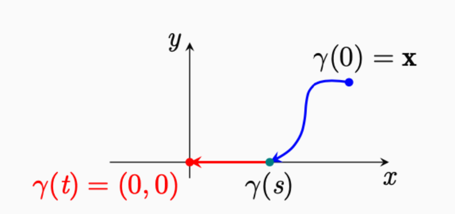
$$\begin{cases} l(\tau) \geq 0 \text{ for } a.e. \tau \in [0,t] \\ l(\tau) = 0 \text{ if } l(\tau) \in \{y > 0\}. \end{cases}$$

Optimal Controls (*Player 1*)

Introduce a condition [10] : there exists a $h \in L^1([0,t],\mathbb{R})$ s.t.

$$\begin{cases} \text{the function} : \tau \mapsto F(\gamma,\eta,l) := l(\tau)G(\frac{(\eta_1-\dot{\gamma}_1)}{l}(\tau)), \\ \text{is integrable on } [0,t], \\ \dot{\gamma}(\tau) = \eta(\tau) \text{ if } l(\tau) = 0 \text{ for } a.e.\tau \in [0,t], \\ |\eta_2(\tau)| = |\dot{\gamma}_2(\tau)| + l(\tau) \text{ if } l(\tau) \neq 0 \text{ for } a.e.\tau \in [0,t]. \end{cases} \tag{H}$$

Denote $N := \{(\gamma,\eta,l)|(\gamma,\eta,l) \text{ satisfies } (H)\}$.



10. G. Barles, H. Ishii, and H. Mitake, ARMA, 2012

Stopping times (*Player 2*)

- A stopping time is a mapping

$$\theta : L^2([0,t];\mathbb{R}^2) \mapsto [0,t],$$

such that for all $s \in [0,t]$, and $\gamma, \hat{\gamma} \in L^2([0,t];\mathbb{R}^2)$, if $\gamma(\tau) = \hat{\gamma}(\tau)$ for a.e. $\tau \in [0,s]$ and $\theta(\tau) \leq s$, then $\theta(\gamma) = \theta(\hat{\gamma})$.

- Θ is the set of all stopping times.
- Define the upper value function :

$$I(x,t) = \sup_{\theta \in \Theta} \inf_{(\gamma,\eta,l) \in N} \Big\{ \int_0^{\theta[\gamma]} L(-\eta(\tau)) + F(\gamma,\eta,l) d\tau + \mathcal{X}_{\{\theta[\gamma]=t\}} g_0(\gamma(t)) \Big\}$$

The upper value function

- *I satisfies the dynamic programming principle*[11]*, i.e. for any* $\sigma \in [0, t]$,

$$I(x,t) = \sup_{\theta \in \Theta} \inf_{(\gamma,\eta,l) \in N} \Big\{ \int_0^{\sigma \wedge \theta[\gamma]} L(-\eta(\tau)) + F(\gamma,\eta,l)(\tau) d\tau + \mathcal{X}_{\{\theta[\gamma] \geq \sigma\}} I(\gamma(\sigma), t-\sigma) \Big\}$$

Theorem

I is continuous on $\mathbb{R}^2 \times \mathbb{R}_+$ and a viscosity solution of (2.1). Moreover, $\lim_{t \to 0+} I(x,t) = g_0(x)$.

11. L. C. Evans and P. E. Souganidis, India. Uni. Math. Jour., 1984

Limits of the phase function

Define

$$v(x,t) = \sup_{\theta\in\Theta} \inf_{(\gamma,\eta,l)\in N} \left\{ \int_0^{\theta[\gamma]} L(-\eta(\tau)) + F(\gamma,\eta,l)(\tau)d\tau \,\middle|\, \gamma(t) = (0,0) \right\}$$

Theorem

$v^*(x,t) \le v \le v_*$ on $\mathbb{R}^2 \times [0,\infty)$. Hence, v is the viscosity solution of (HJ).

Brief Proof.

- *Step 1. Show that v^* and v_* are sub-solution and super-solution of (HJ), respectively.*
- *Step 2. Prove $v_*(x,t) \ge v(x,t)$.*
- *Step 3. Prove $v^*(x,t) \le v(x,t)$.*

Limits of the phase function

Define

$$v(x,t) = \sup_{\theta\in\Theta} \inf_{(\gamma,\eta,l)\in N} \left\{ \int_0^{\theta[\gamma]} L(-\eta(\tau)) + F(\gamma,\eta,l)(\tau)d\tau \middle| \gamma(t) = (0,0) \right\}$$

Theorem

$v^*(x,t) \le v \le v_*$ on $\mathbb{R}^2 \times [0,\infty)$. Hence, v is the viscosity solution of (HJ).

Brief Proof.

- *Step 1. Show that v^* and v_* are sub-solution and super-solution of (HJ), respectively.*
- *Step 2. Prove $v_*(x,t) \ge v(x,t)$.*
- *Step 3. Prove $v^*(x,t) \le v(x,t)$.*

Limits of the phase function

Define a payoff

$$J(x,y,t) = \inf_{(\gamma,\eta,l)\in N}\left\{ \int_0^{\theta[\gamma]} L(-\eta(\tau)) + F(\gamma,\eta,l)(\tau)d\tau \middle| \gamma(t) = (0,0)\right\},$$

Then, we can prove $v(x,y,t) = \max\{0, J(x,y,t)\}$ by verifying the Freidlin's condition [12] :

$$J(x,y,t) = \inf_{(\gamma,\eta,l)\in N}\left\{ \int_0^{\theta[\gamma]} L(-\eta(\tau)) + F(\gamma,\eta,l)(\tau)d\tau \middle| \right.$$
$$\left. \gamma(t) = (0,0), (\gamma(\tau), t-\tau) \in P \right\},$$

for $(x,y,t) \in \partial P$, where $P := \{J > 0\}$.

12. M. I. Freidlin, Princeton University Press, 1985

Limits of the phase function

Theorem

$$J(x,y,t) = \varphi^*(x,y,t) - t,$$

where

$$\varphi^*(x,y,t) = \min_{s\geq 0}\left\{\frac{x^2}{4(t+as)} + \frac{(|y|+s)^2}{4t}\right\}.$$

The result was first proved by X.F. Chen, J.F. He, and X.F. Wang. [13]

13. The asymptotic propagation speed of the Fisher-KPP equation with effective boundary condition on a road, preprint

Limits of the phase function

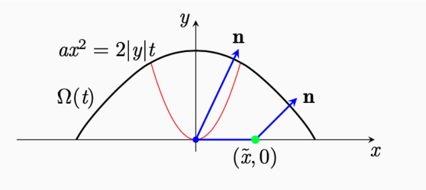
Brief proof :

- *Step 1.* φ^* *is the lower-bound of* J*, i.e.*

$$J(x,y,t) \geq \frac{x^2}{4(t+as)} + \frac{(|y|+s)^2}{4t} - t \geq \varphi^*.$$

- *Step 2. Find control triple to attain the lower-bound, i.e.*

$$J(x,y,t) = \varphi^*.$$



Limits of the phase function

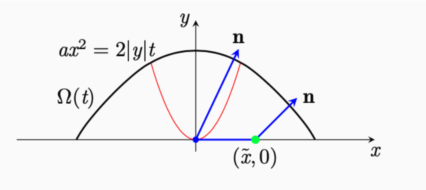
Brief proof :

- *Step 1.* φ^* *is the lower-bound of* J*, i.e.*

$$J(x,y,t) \geq \frac{x^2}{4(t+as)} + \frac{(|y|+s)^2}{4t} - t \geq \varphi^*.$$

- *Step 2. Find control triple to attain the lower-bound, i.e.*

$$J(x,y,t) = \varphi^*.$$



Ongoing : multiple roads

- The full model :

$$\begin{cases} u_t - \nabla \cdot (\tilde{\sigma} \nabla u) = u(1-u), & (x,y) \in \mathbb{R}^2, t > 0, \\ u(x,y,0) = \varphi(x,y), & (x,y) \in \mathbb{R}^2, \\ 0 \le \varphi \le 1, \;\; \varphi \not\equiv 0, \;\; \varphi \text{ compactly supported} & \end{cases}$$

where

$$\tilde{\sigma} = \begin{cases} \sigma, & \text{if } y \in (1, 1+\delta) \cup (-1-\delta, -1), \\ 1, & \text{else.} \end{cases}$$

- The full model :

$$\begin{cases} u_t - \nabla \cdot (\tilde{\sigma} \nabla u) = u(1-u), & (x,y) \in \mathbb{R}^2, t > 0, \\ u(x,y,0) = \varphi(x,y), & (x,y) \in \mathbb{R}^2, \\ 0 \leq \varphi \leq 1, \quad \varphi \not\equiv 0, \quad \varphi \text{ compactly supported} \end{cases}$$

where

$$\tilde{\sigma} = \begin{cases} \sigma, & \text{if } y \in (1, 1+\delta) \cup (-1-\delta, -1), \\ 1, & \text{else.} \end{cases}$$

THANK YOU !