Effective Boundary Conditions for the Fisher-KPP Equations

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3. EBCs for the Fisher-KPP Equation
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5. Future Works

1 Introduction

2 EBCs for the Heat Equation

3 EBCs for the Fisher-KPP Equation

4 EBCs for the System

5 Future Works

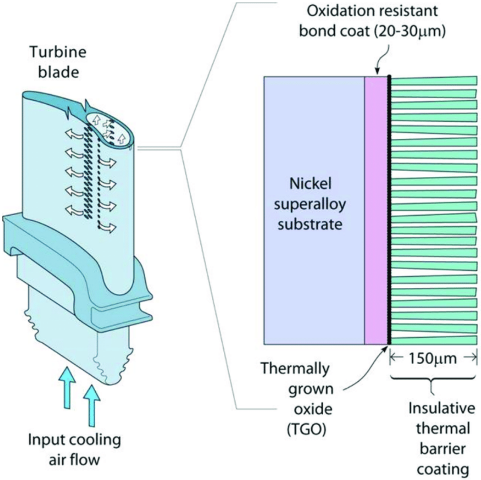

Figure – Turbine Engine Blades (from A. Aabid, S.A. Khan, 2008).

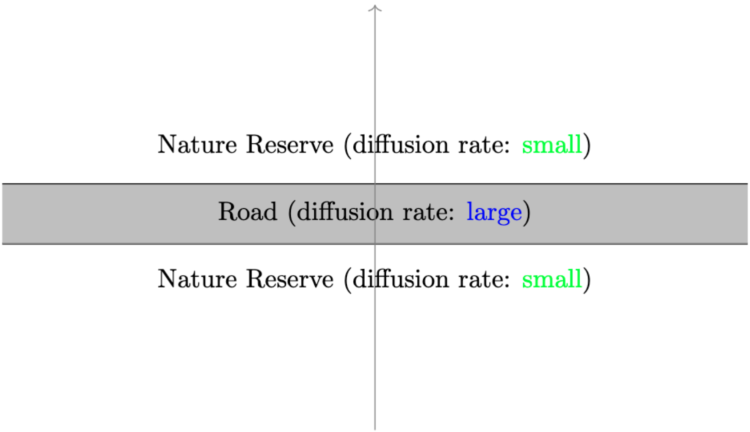

Figure – Nature Reserve

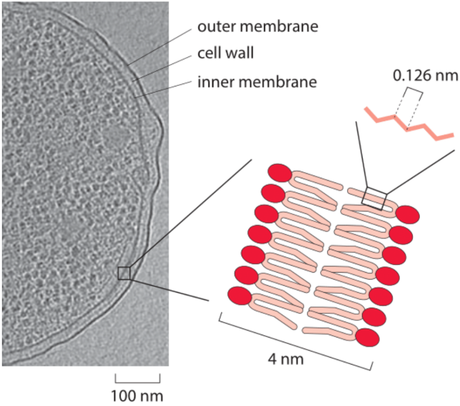

Figure – Cells (from A. Briegel et al., 2009).

- Domain contains a thin component :
  - thermal barrier coatings for blades ;
  - a road for nature reserve ;
  - membrane for cells.
- Diffusion tensor on different components is drastically different :
  - in coating model, diffusion tensor is small ;
  - in nature reserve model, diffusion rate is large ;
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- Issues :
  - The multi-scale in size and different diffusion tensors lead to computational difficulty ;
  - It is hard to see the effect of the thin component ;
- Resolution :
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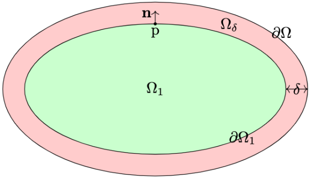

Figure – $\Omega = \Omega_1 \cup \overline{\Omega}_\delta \subset \mathbb{R}^n$. $\partial\Omega \to \partial\Omega_1$ as $\delta \to 0$.

- Ω_δ : uniformly thick with thickness δ.
- Ω_1 : fixed with $\partial\Omega_1 (= \Gamma) \in C^2$.
- Use curvilinear coordinates (s, r) in Ω_δ :

$$x = p(s) + r\mathbf{n}(s), \quad \forall x \in \Omega_\delta,$$

$p-$the projection of x onto $\partial\Omega_1$, $\mathbf{n}-$unit outer normal vector of Ω_1, $r-$ distance from x to $\partial\Omega_1$, $s = (s_1, ..., s_{n-1})-$ local coordinates.

For any fixed $T > 0$, $u := u(x,t)$ satisfies

$$\begin{cases} u_t - \nabla \cdot (A(x)\nabla u) = f(x,t), & x \in \Omega, t \in (0,T), \\ u = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

- $u_0 \in L^2(\Omega),\ f \in L^2(\Omega \times (0,T))$;
- Transmission conditions on $\partial\Omega_1 \times (0,T)$:

$$u_1 = u_\delta,\ k\frac{\partial u_1}{\partial \mathbf{n}} = A(x)\nabla u_\delta \cdot \mathbf{n},$$

$u_1,\ u_\delta-$ the restrictions of u on $\Omega_1 \times (0,T)$ and $\Omega_\delta \times (0,T)$;

Goal : given some assumptions on the diffusion tensor $A(x)$, $u \to$ some v as $\delta \to 0$ with some EBCs satisfied.

History

- In 1959, Carlaw and Jaeger derived EBCs in some simple cases in their classic book *Conduction of Heat in Solids* ;
- In 1974, Sanchez-Palencia studied elliptic and heat equations with thin diamond-shaped inclusions ;
- In 1980, Brezis, Caffarelli and Friedman studied the elliptic problem in both interior and boundary reinforcement cases ;
- Lots of work on elastic equations, electromagnetic equations, nonlinear diffusion equations, etc.
- Among these work, the diffusion tensor $A(x)$ is isotropic in the layer.

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- In 2012, Chen, Pond and Wang studied the optimally aligned coating in 2-dimensional case to derive new EBCs ;
- In 2017, Li and Wang used the idea of EBCs to derive the model about the effect of fast diffusion on the road, which is different from Berestycki's model ;
- In 2021, Li, Su, Wang and Wang derived the Bulk-Surface model by using the ideal of EBCs ;
- In 2023, Chen, He and Wang studied the effect of EBCs on the propagation speed of the Fisher-KPP equation.

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In our case, let $n = 3$, and

$$A(x) = \begin{cases} kI_{3\times 3}, & x \in \Omega_1, \\ (a_{ij}(x))_{3\times 3}, & x \in \Omega_\delta. \end{cases}$$

- $k > 0$ constant, $(a_{ij})_{3\times 3}$: anisotropic and positive-definite.
- Ω_δ is "optimally aligned"[1] : for any $x \in \Omega_\delta$, the normal vector $\mathbf{n}(p)$ is always an eigenvector of $A(x)$.
- $A(x)$ satisfies :

$$A(x)\mathbf{n}(p) = \sigma\mathbf{n}(p), A(x)\boldsymbol{\tau}_1(p) = \mu_1\boldsymbol{\tau}_1(p), A(x)\boldsymbol{\tau}_2(p) = \mu_2\boldsymbol{\tau}_2(p),$$

  - $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$– two eigenvectors of $A(x)$;
  - σ– "normal diffusion rate" ;
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- *Case 1.* $\mu_1 = \mu_2$.
  Assume $A(x)$ satisfies

$$A(x)\mathbf{n}(p) = \sigma\mathbf{n}(p), \quad A(x)\mathbf{s}(p) = \mu\mathbf{s}(p), \quad \forall x \in \Omega_\delta,$$

  where $(\sigma, \mu) = (\sigma(\delta), \mu(\delta))$; $\mathbf{n}(p)-$ unit outer normal vector of Ω_1, $\mathbf{s}(p)-$ arbitrary tangent vector at p on $\partial\Omega_1$.

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  Assume $\partial\Omega_1 = \mathbb{T}^2$, and $A(x)$ satisfies

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Denote $Q_T := \Omega \times (0,T)$ and $S_T := \partial\Omega \times (0,T)$.

- $W_2^{1,0}(Q_T) = \{u \in L^2(Q_T) \big| \nabla u \in L^2(Q_T)\}$ and $W_{2,0}^{1,0}(Q_T)$ is the closure of C^∞ functions vanishing near $\overline{S}_T$ in $W_2^{1,0}(Q_T)$-norm.
- $V_{2,0}^{1,0}(Q_T) = W_{2,0}^{1,0}(Q_T) \cap C\left([0,T]; L^2(\Omega)\right)$.

Definition

u is a weak solution of the heat equation, if $u \in V_{2,0}^{1,0}(Q_T)$ and

$$\mathcal{A}[u,\xi] = -\int_\Omega u_0(x)\xi(x,0)dx + \int_{Q_T} (A(x)\nabla u)\cdot\nabla\xi - u\xi_t - f\xi dtdx$$
$$=0$$

for any $\xi \in W_{2,0}^{1,1}(Q_T)$ satisfying $\xi = 0$ at $t = T$.

(Recall $A(x)\mathbf{n}(p) = \sigma\mathbf{n}(p), A(x)\mathbf{s}(p) = \mu\mathbf{s}(p), \forall x \in \Omega_\delta$.)

Theorem

[a] *Let*

$$\lim_{\delta\to 0}\frac{\sigma}{\delta} = \alpha \in [0,\infty] \text{ and } \lim_{\delta\to 0}\sigma\mu = \gamma \in [0,\infty].$$

As $\delta \to 0$, $u \to v$ *in* $C([0,T];L^2(\Omega))$, *where* $v := v(x,t)$ *is the unique solution of the effective problem*

$$\begin{cases} v_t - k\Delta v = f(x,t), & x \in \Omega_1, t \in (0,T), \\ v(x,0) = u_0(x), & x \in \Omega_1, \end{cases}$$

with the EBCs listed in the table below :

a. X. Geng, preprint, 2023

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0, \infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu} \to \gamma \in (0, \infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\gamma/\alpha}[v]$	$v = 0$
$\sigma\mu \to \infty$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_\Gamma (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$	$v = 0$

- $\mathbf{n}$– unit outer normal vector of Γ.
- ∇_Γ– the surface gradient operator.
- $\mathcal{J}_D^{\gamma/\alpha}$– a Dirichlet-to-Neumann mapping.

Given a smooth function $g(s)$ and $H \in (0, \infty)$, define

$$\mathcal{J}_D^H[g] := \Psi_R(s, 0),$$

where $\Psi := \Psi(s, R)$ is the unique solution of

$$\begin{cases} \Psi_{RR} + \Delta_\Gamma \Psi = 0, & \Gamma \times (0, H), \\ \Psi(s, 0) = g(s), & \Psi(s, H) = 0. \end{cases}$$

Moreover,

$$\mathcal{J}_D^H[g](s) = -\sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n} e_n(s) g_n}{\tanh(\sqrt{\lambda_n} H)}, \mathcal{J}_D^\infty = \lim_{H \to \infty} \mathcal{J}_D^H = -(-\Delta_\Gamma)^{1/2}.$$

$\lambda_n, e_n(s)$– the eigenvalues and the corresponding eigenfunctions of the Laplacian-Beltrami $-\Delta_\Gamma$, and $g_n := \langle e_n, g \rangle$.

Outline of the proof :

- Step 1. Existence and uniqueness of the solution of heat equation.
- Step 2. Energy estimates for the solution of heat equation and then apply the Arzela-Ascoli theorem to show that after passing to a subsequence of δ, $u \to v$.
- Step 3. Such v is a weak solution of the effective problem.
- Step 4. Uniqueness of the solution of the effective problem to ensure the convergence without passing to a subsequence..

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Step 3. For given ξ that is the test function of heat equation, $\xi \in C^\infty(\overline{\Omega}_1 \times (0,T))$.

Take a new test function [2]

$$\bar{\xi}(x,t) = \begin{cases} \xi(x,t), & \Omega_1, \\ \psi(x,t), & \overline{\Omega}_\delta, \end{cases}$$

where $\psi := \psi(s,r,t)$ is the unique solution of

$$\begin{cases} \sigma\psi_{rr} + \mu\Delta_\Gamma\psi = 0, & \Gamma \times (0,\delta), \\ \psi(s,0,t) = \xi(s,0,t) & \psi(s,\delta,t) = 0. \end{cases}$$

2. X. Chen, C. Pond, and X. Wang, ARMA, 2012

- By the weak solution of heat equation, it holds

$$\int_0^T \int_{\Omega_1} k\nabla\xi \cdot \nabla u dxdt - \int_\Omega u_0(x)\overline{\xi}(x,0)dx - \int_0^T \int_\Omega (u\overline{\xi_t} + f\overline{\xi})dxdt$$
$$= -\int_0^T \int_{\Omega_\delta} \nabla\psi \cdot A(x)\nabla u dxdt.$$

- EBCs arise on the right-hand side.

Remark :

- If the layer is of interior inclusion, EBCs are also derived [3].

3. X. Geng, CPAA, 2023

(Recall $A(x)\mathbf{n}(p) = \sigma \mathbf{n}(p), A(x)\boldsymbol{\tau}_1(p) = \mu_1 \boldsymbol{\tau}_1(p), A(x)\boldsymbol{\tau}_2(p) = \mu_2 \boldsymbol{\tau}_2(p)$.)

Theorem

[a] *Suppose* $\Gamma = \mathbb{T}^2 = \Gamma_1 \times \Gamma_2$ *and* $\mu_1 > \mu_2$.
Let

$$\lim_{\delta \to 0} \frac{\mu_2}{\mu_1} = c \in [0,1], \quad \lim_{\delta \to 0} \frac{\sigma}{\delta} = \alpha \in [0,1],$$
$$\lim_{\delta \to 0} \sigma \mu_i = \gamma_i \in [0,\infty], \quad \lim_{\delta \to 0} \mu_i \delta = \beta_i \in [0,\infty], \quad i = 1,2.$$

a. X. Geng, preprint, 2023

EBCs for the Heat Equation

Case 2 : $\mu_1 \neq \mu_2$

Theorem

(i) If $c \in (0,1]$, then as $\delta \to 0$, $u \to v$ in $C([0,T];L^2(\Omega_1))$, where $v := v(x,t)$ is the solution of the effective problem with the EBCs listed in the following table :

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu_1 \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^{\gamma_1/\alpha}[v]$	$v = 0$
$\sigma\mu_1 \to \infty$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_\Gamma (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$	$v = 0$

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(ii) If $c = 0$, $\lim_{\delta\to 0} \delta^2\mu_1/\mu_2 = 0$, *then* $u \to v$ *in* $C([0,T]; L^2(\Omega_1))$, *where* v *is the solution of the effective problem with the EBCs listed in the table :*

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
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$\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^{\gamma_1/\alpha}[v]$	$v = 0$
$\sigma\mu_1 \to \infty, \sigma\mu_2 \to 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \frac{\partial v}{\partial \mathbf{n}} = 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(\frac{\partial v}{\partial \mathbf{n}} + \alpha v\right) = 0$	$v = 0$
$\sigma\mu_1 \to \infty$, $\sqrt{\sigma\mu_2} \to \gamma_2 \in (0,\infty)$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^{\infty}[v]\right) = 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^{\gamma_2/\alpha}[v]\right) = 0$	$v = 0$
$\sigma\mu_1 \to \infty, \sigma\mu_2 \to \infty$	$\nabla_\Gamma v = 0$, $\int_{\Gamma} \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_{\Gamma} \frac{\partial v}{\partial \mathbf{n}} = 0$	$v = 0$

- If $\frac{\sigma}{\delta} \to \infty$ as $\delta \to 0$, then the tangential diffusion rate has no influence (i.e. the EBC is always $v = 0$).
- If $\sigma\mu_1 \to \gamma_1 \in [0, \infty)$, then μ_2 has no influence on EBCs.
- Similarly, for smooth g and for $H \in (0, \infty)$, define

$$\mathcal{K}_D^H[g](s) := \Phi_R(s, 0),$$

where Φ is the unique bounded solution of

$$\begin{cases} \Phi_{RR} + \Phi_{s_1 s_1} + c\Phi_{s_2 s_2} = 0, & \Gamma \times (0, H), \\ \Phi(s, 0) = g(s), & \Phi(s, H) = 0. \end{cases}$$

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- If $\frac{\sigma}{\delta} \to \infty$ as $\delta \to 0$, then the tangential diffusion rate has no influence (i.e. the EBC is always $v = 0$).
- If $\sigma\mu_1 \to \gamma_1 \in [0, \infty)$, then μ_2 has no influence on EBCs.
- Similarly, for smooth g and for $H \in (0, \infty)$, define

$$\mathcal{K}_D^H[g](s) := \Phi_R(s, 0),$$

where Φ is the unique bounded solution of

$$\begin{cases} \Phi_{RR} + \Phi_{s_1 s_1} + c\Phi_{s_2 s_2} = 0, & \Gamma \times (0, H), \\ \Phi(s, 0) = g(s), & \Phi(s, H) = 0. \end{cases}$$

- $\Lambda_D^H[g](s) := \Phi_R^0(s,0)$, where Φ^0 is the unique bounded solution of

$$\begin{cases} \Phi^0_{RR} + \Phi^0_{s_1 s_1} = 0, & \Gamma \times (0,H), \\ \Phi^0(s,0) = g(s), & \Phi^0(s,H) = 0. \end{cases}$$

- $\mathcal{D}_D^H[g](s_2) := \Phi_R(s_2,0)$, where Φ is the unique bounded solution of

$$\begin{cases} \Phi_{RR} + \Phi_{s_2 s_2} = 0, & \Gamma_2 \times (0,H), \\ \Phi(s_2,0) = g(s_2), & \Phi(s_2,H) = 0. \end{cases}$$

- The proof is similar to that in Case 1.
- Step 3. Construct an auxiliary function ϕ by defining

$$\begin{cases} \sigma\phi_{rr} + \mu_1\phi_{s_1s_1} + \mu_2\phi_{s_2s_2} = 0, & \Gamma \times (0,\delta), \\ \phi(s,0,t) = \xi(s,0,t), & \phi(s,\delta,t) = 0. \end{cases}$$

- Let $r = R\sqrt{\sigma/\mu_1}$ and suppress the time dependence, leading to

$$\begin{cases} \Phi^\delta_{RR} + \Phi^\delta_{s_1s_1} + \frac{\mu_2}{\mu_1}\Phi^\delta_{s_2s_2} = 0, & \Gamma \times (0,h_1), \\ \Phi^\delta(s,0) = \xi(s,0,t), & \Phi^\delta(s,h_1) = 0, \end{cases}$$

where $h_1 = \delta\sqrt{\sigma/\mu_1}$ and $\Phi^\delta(s,R) := \phi(s,R\sqrt{\sigma/\mu_1},t)$.

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where $h_1 = \delta\sqrt{\sigma/\mu_1}$ and $\Phi^\delta(s,R) := \phi(s, R\sqrt{\sigma/\mu_1}, t)$.

(Recall $A(x)\mathbf{n}(p) = \sigma\mathbf{n}(p), A(x)\mathbf{s}(p) = \mu\mathbf{s}(p), \forall x \in \Omega_\delta$.)

Theorem

Let $\sigma\mu \to 0$ *and* $\frac{\sigma}{\delta} \to \alpha \in [0,\infty)$ *as* $\delta \to 0$. *Thus, the EBC is*

$$\frac{\partial v}{\partial \boldsymbol{n}} + \alpha v = 0.$$

Under some assumptions on $\partial\Omega_1, u_0, f$, *the following holds.*
(i) If $\alpha \in (0,\infty)$, *then*

$$||u(\cdot,t) - v(\cdot,t)||^2_{L^2(\Omega_1)} \leq C\left(\left|\frac{\sigma}{\delta} - \alpha\right| + \sqrt{\delta} + \sqrt{\sigma\mu}\right).$$

(ii) If $\alpha = 0$, *then*

$$||u(\cdot,t) - v(\cdot,t)||^2_{L^2(\Omega_1)} \leq C\left(\sqrt{\sigma\mu t e^t} + \sqrt{\delta t e^t} + \frac{\sigma t e^t}{\delta}\right).$$

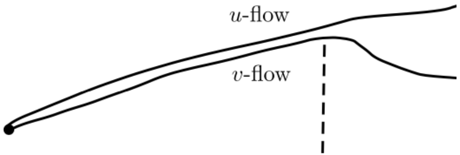
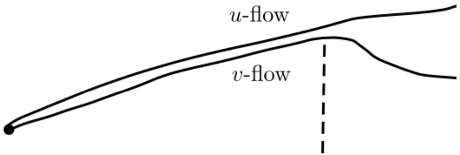


Figure – An illustration of $||u(\cdot, t) - v(\cdot, t)||_{L^2(\Omega_1)}$ as $t \to \infty$.

- Question : what is the maximal interval that keeps u and v close ?
- Answer : consider the steady state of u and v.
- A typical example : the EBC is a Neumann condition.



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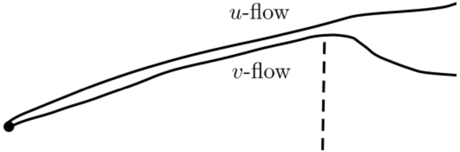

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(Recall $A(x)\mathbf{n}(p) = \sigma\mathbf{n}(p), A(x)\mathbf{s}(p) = \mu\mathbf{s}(p), \forall x \in \Omega_\delta$.)
For any fixed $T > 0$, $u := u(x,t)$ satisfies

$$\begin{cases} u_t - \nabla \cdot (A(x)\nabla u) = f(u), & x \in \Omega, t \in (0,T), \\ u = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

- $0 \le u_0 \in L^\infty(\Omega)$, $f(u) = u(1-u)$.
- Transmission conditions :

$$u_1 = u_\delta, \; k\frac{\partial u_1}{\partial \mathbf{n}} = A(x)\nabla u_\delta \cdot \mathbf{n}.$$

$u_1, u_\delta-$ the restrictions of u on $\Omega_1 \times (0,T)$ and $\Omega_\delta \times (0,T)$.

The derivation of EBCs is similar to that in the heat equation

Theorem

As $\delta \to 0$, u satisfies

$$\max_{0 \le t \le \infty} ||u(\cdot,t) - v(\cdot,t)||_{L^2(\Omega_1)} \to 0,$$

where v is the solution of the effective problem with any EBC as follows.

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu} \to \gamma \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\gamma/\alpha}[v]$	$v = 0$
$\sigma\mu \to \infty$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_\Gamma (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$	$v = 0$

The idea is to consider the steady state of u and v.

- Consider

$$\begin{cases} -\nabla \cdot (A(x)\nabla U) = U(1-U), & x \in \Omega, \\ U = 0, & x \in \partial\Omega, \end{cases}$$

  where $U := U(x)$ is the unique positive solution.

- $V := V(x)$ is the unique positive solution of

$$-k\Delta V = V(1-V),\ x \in \Omega_1,$$

  with the EBCs listed in the above table (with v replaced by V).

Outline of the proof :

- $||u(\cdot,t) - U||_{L^2(\Omega)}$ is decreasing in t.
- For any $t \geq T_\varepsilon$,

$$
\begin{aligned}
&||u(\cdot,t) - U||_{L^2(\Omega_1)} \\
&\leq ||u(\cdot,t) - U||_{L^2(\Omega)} \\
&\leq ||u(\cdot,T_\varepsilon) - v(\cdot,T_\varepsilon)||_{L^2(\Omega_1)} + ||v(\cdot,T_\varepsilon) - V||_{L^2(\Omega_1)} \\
&\quad + ||U - V||_{L^2(\Omega_1)} + ||u(\cdot,T_\varepsilon) - U||_{L^2(\Omega_2)} \\
&\leq C\varepsilon.
\end{aligned}
$$

- Finally, for a small $\delta > 0$,

$$\begin{aligned}
&\max_{t\in[T_\varepsilon,\infty]}||u(\cdot,t)-v(\cdot,t)||_{L^2(\Omega_1)}\\
&\le \max_{t\in[T_\varepsilon,\infty]}||u(\cdot,t)-U||_{L^2(\Omega_1)}+\max_{t\in[T_\varepsilon,\infty]}||v(\cdot,t)-V||_{L^2(\Omega_1)}\\
&\quad +||U-V||_{L^2(\Omega_1)}\\
&\le ||u(\cdot,T_\varepsilon)-U||_{L^2(\Omega_1)}+\max_{t\in[T_\varepsilon,\infty]}||v(\cdot,t)-V||_{L^2(\Omega_1)}+||U-V||_{L^2(\Omega_1)}\\
&\le C\varepsilon.
\end{aligned}$$

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Denote $\mathbb{R}^2_- := \{(x,y) : x \in \mathbb{R}, y < 0\}$ and $\Gamma_1 := \{(x,y) : x \in \mathbb{R}, y = 0\}$.

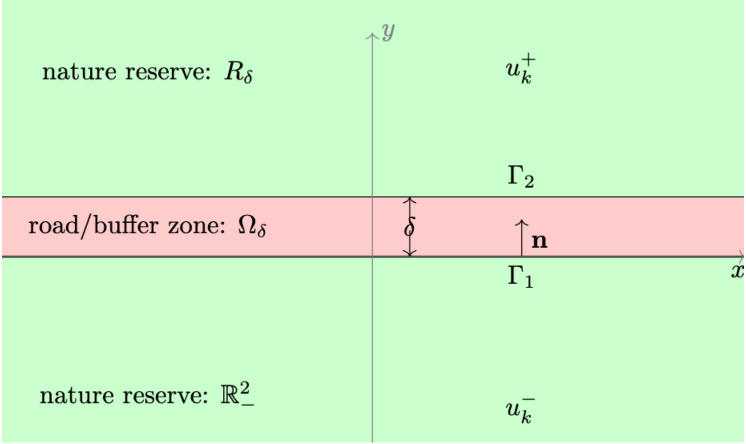


Figure – $\Omega_\delta \subset \mathbb{R}^2$ is uniformly thick with thickness δ. $\Gamma_2 \to \Gamma_1$ as $\delta \to 0$.

Consider the coupled Fisher-KPP equations (the Lotka-Volterra competition diffusion system) in $\mathbb{R}^2$

$$\begin{cases} \partial_t u_1 - \nabla \cdot (D_1(x,y)\nabla u_1) = f_1(u_1,u_2), & (x,y) \in \mathbb{R}^2, t>0, \\ \partial_t u_2 - \nabla \cdot (D_2(x,y)\nabla u_2) = f_2(u_1,u_2), & (x,y) \in \mathbb{R}^2, t>0, \\ (u_1,u_2)(x,y,0) = (u_{1,0},u_{2,0})(x,y), & (x,y) \in \mathbb{R}^2, \end{cases}$$

where

- $u_1 := u_1(x,y,t), u_2 := u_2(x,y,t)$,
- $f_1(u_1,u_2) = r_1 u_1(1-u_1-b_1u_2)$, $f_2(u_1,u_2) = r_2u_2(1-u_2-b_2u_1)$,
- $b_1, b_2 \in (0,1)$, the initial value $u_{k,0}(k=1,2)$ satisfying

$$\begin{cases} 0 \le u_{k,0} \le 1, u_{k,0} \not\equiv 0, \\ u_{k,0} \quad \text{are } C^\infty\text{– smooth, and compactly supported.} \end{cases}$$

Let

$$D_k(x,y) = \begin{cases} a_{ij}^k(x,y), & \text{if } y \in (0,\delta), \\ d_k, & \text{otherwise.} \end{cases}$$

- $d_k > 0$ are constants, $a_{ij}^k(x,y)$ is positive-definite and satisfies the optimally aligned condition in the road :

$$D_k(x,y)\mathbf{n}(x) = \sigma_k \mathbf{n}(x), D_k(x,y)\mathbf{s}(x) = \mu_k \mathbf{s}(x), \quad \forall y \in (0,\delta),$$

where $\mathbf{n}(x) = (0,1), \mathbf{s}(x) = (1,0)$.

Motivated by the work of Li and Wang [4], we have the existence and uniqueness of the system.

Theorem

For any fixed $T > 0$, the system admits a unique bounded solution

$$u_k \in V_2^{1,1}(\mathbb{R}^2 \times (0,T)), \quad k = 1, 2.$$

Moreover, $0 \le u_k \le M$ for some M independent of δ, and

$$u_k \in C_{loc}^{\infty}\left(\overline{\Omega}_\delta \times (0,T)\right) \cap C_{loc}^{\infty}\left(\overline{R}_\delta \times (0,T)\right) \cap C_{loc}^{\infty}\left(\overline{\Omega}_- \times (0,T)\right).$$

4. H. Li and X. Wang, Nonlinearity, 2017

Theorem

For any fixed $T > 0$, *and* $k = 1, 2$, *let*

$$\lim_{\delta\to 0} \sigma_k \mu_k = \gamma_k \in [0, \infty], \ \lim_{\delta\to 0} \frac{\sigma_k}{\delta} = \alpha_k \in [0, \infty], \ \lim_{\delta\to 0} \mu_k \delta = \beta_k \in [0, \infty].$$

Then $(u_1, u_2) \to (v_1, v_2)$ *in* $C\left([0, T], L^2_{loc}(\mathbb{R}^2)\right) \times C\left([0, T], L^2_{loc}(\mathbb{R}^2)\right)$ *as* $\delta \to 0$, *where* (v_1, v_2) *is the solution of the effective system*

$$\begin{cases} \partial_t v_1 - d_1 \Delta u_1 = r_1 v_1 (1 - v_1 - b_1 v_2), & x \in \mathbb{R}, y \neq 0, t > 0, \\ \partial_t v_2 - d_2 \Delta v_2 = r_2 v_2 (1 - v_2 - b_2 v_1), & x \in \mathbb{R}, y \neq 0, t > 0, \\ (v_1, v_2)(x, y, 0) = (u_{1,0}, u_{2,0})(x, y), & (x, y) \in \mathbb{R}^2, \end{cases}$$

with the EBCs listed below.

Case 1. $\frac{\sigma_k}{\delta} \to 0$ as $\delta \to 0$.

As $\delta \to 0$	$\gamma_k = 0$	$\gamma_k \in (0, \infty)$	$\gamma_k = \infty$
$\beta_k \in [0, \infty)$	$\frac{\partial v_k^-}{\partial y} = \frac{\partial v_k^+}{\partial y} = 0$	$-------$	$-------$
$\beta_k = \infty$	$\frac{\partial v_k^-}{\partial y} = \frac{\partial v_k^+}{\partial y} = 0$	$\frac{\partial v_k^-}{\partial y} = \gamma_k \mathcal{J}_1^\infty[v_k^-]$, $\frac{\partial v_k^+}{\partial y} = -\gamma_k \mathcal{J}_1^\infty[v_k^+]$	$v_k^- = v_k^+ = 0$

- The dash lines mean such cases do not exist.
- v_k^- and v_k^+ : the restrictions of v_k on $\mathbb{R}^2_- \times (0, T)$ and $\mathbb{R}^2_+ \times (0, T)$.

Case 2. $\frac{\sigma_k}{\delta} \to \alpha_k \in (0,\infty)$ as $\delta \to 0$.

As $\delta \to 0$	$\gamma_k = 0$	$\gamma_k \in (0,\infty)$	$\gamma_k = \infty$
$\beta_k = 0$	$\frac{\partial v_k^-}{\partial y} = \frac{\partial v_k^+}{\partial y}$, $d_k \frac{\partial v_k^-}{\partial y} = \alpha_k (v_k^+ - v_k^-)$	$------$	$------$
$\beta_k \in (0,\infty)$	$------$	$d_k \frac{\partial v_k^-}{\partial y} = \gamma_k \mathcal{J}_1^{\beta_k/\gamma_k}[v_k^-] - \gamma_k \mathcal{J}_2^{\beta_k/\gamma_k}[v_k^+]$, $d_k \frac{\partial v_k^+}{\partial y} = \gamma_k \mathcal{J}_2^{\beta_k/\gamma_k}[v_k^-] - \gamma_k \mathcal{J}_1^{\beta_k/\gamma_k}[v_k^+]$	$------$
$\beta_k = \infty$	$------$	$------$	$v_k^- = v_k^+ = 0$

Case 3. $\frac{\sigma_k}{\delta} \to \infty$ and $\sigma_k \delta^3 \to 0$ as $\delta \to 0$.

As $\delta \to 0$	$\gamma_k \in [0, \infty)$	$\gamma_k = \infty$
$\beta_k = 0$	$v_k^- = v_k^+, \frac{\partial v_k^-}{\partial y} = \frac{\partial v_k^+}{\partial y}$	$v_k^- = v_k^+, \frac{\partial v_k^-}{\partial y} = \frac{\partial v_k^+}{\partial y}$
$\beta_k \in (0, \infty)$	$- - - - - --$	$v_k^- = v_k^+$, $d_k(\frac{\partial v_k^-}{\partial y} - \frac{\partial v_k^+}{\partial y}) = \beta_k \partial_{xx} v_k^+$
$\beta_k = \infty$	$- - - - - --$	$v_k^- = v_k^+ = 0$

- The condition $\sigma_k \delta^3 \to 0$ can be removed if $\frac{\mu_k}{\sigma_k} \nrightarrow 0$.

- $\mathcal{J}_1^{\beta/\gamma}, \mathcal{J}_2^{\beta/\gamma}$ – the Dirichlet-to-Neumann mapping.
- For $H \in (0, \infty)$ and smooth g on $\mathbb{R}$, define

$$\mathcal{J}_1^H[g] := \Psi_Y(x, 0) \quad \text{and} \quad \mathcal{J}_2^H[g] := \Psi_Y(x, H),$$

where Ψ is the unique solution of

$$\begin{cases} \Psi_{YY} + \Psi_{xx} = 0, & \mathbb{R} \times (0, H), \\ \Psi(x, 0) = g(x), & \Psi(x, H) = 0. \end{cases}$$

- Moreover,

$$\mathcal{J}_1^\infty[g] = \lim_{H \to \infty} \mathcal{J}_1^H[g] := -(-\partial_{xx})^{1/2} g, \; \mathcal{J}_2^\infty[g] = \lim_{H \to \infty} \mathcal{J}_2^H[g] = 0,$$

where $(-\partial_{xx})^{1/2} g$ is the fractional Laplacian of order $1/2$.

Consider the Lotka-Volterra competition diffusion system :

$$\begin{cases} \partial_t v_1 - \Delta v_1 = v_1(1 - v_1 - b_1 v_2), & x \in \mathbb{R}, y \neq 0, t > 0, \\ \partial_t v_2 - d\Delta v_2 = r v_2(1 - v_2 - b_2 v_1), & x \in \mathbb{R}, y \neq 0, t > 0, \\ [v_1] = 0, [(v_1)_y] = -2a_1 (v_1)_{xx}, & x \in \mathbb{R}, y = 0, t > 0, \\ [v_2] = 0, [d(v_2)_y] = -2a_2 (v_2)_{xx}, & x \in \mathbb{R}, y = 0, t > 0, \\ (v_1, v_2)(x, y, 0) = (v_{10}, v_{20})(x, y), & (x, y) \in \mathbb{R}^2, \end{cases}$$

where

- $[v_1]\Big|_{y=0} := v_1(x, 0+, t) - v_1(x, 0-, t)$;
- $dr > 1, b_1, b_2 \in (0, 1)$ and $a_1, a_2 \in (0, \infty)$ with $a_1 \ll a_2$;
- the initial value (v_{10}, v_{20}) satisfies

$$\begin{cases} 0 \leq v_{10} \leq 1, v_{10} \not\equiv 0, \\ 0 \leq v_{20} \leq 1, v_{20} \not\equiv 0, \\ v_{10}, v_{20} \text{ are compactly supported.} \end{cases}$$

If no competition effect, the Fisher-KPP equation with a Wenztel-type boundary condition (can seen as an EBC) reads as

$$\begin{cases} \partial_t v - \Delta v = v(1-v), & x \in \mathbb{R}, y \neq 0, t > 0, \\ [v] = 0, [v_y] = -2av_{xx}, & x \in \mathbb{R}, y = 0, t > 0, \\ v(x,y,0) = v_0(x,y), & (x,y) \in \mathbb{R}^2. \end{cases}$$

- This model was derived by Li and Wang[5].
- The spreading speed and shape was studied by Chen, He and Wang[6].

5. H. Li and X. Wang, Nonlinearity, 2017
6. X. Chen, J. He and X. Wang, ARMA, 2023

Theorem (Chen, He and Wang)

For each $\nu \in (0,1)$,

$$\lim_{t\to\infty} \|v(\cdot,t)\|_{L^\infty(\Omega^c(t))} = 0, \quad \lim_{t\to\infty} \|v(\cdot,t) - 1\|_{L^\infty(\Omega(\nu t))} = 0,$$

where $\Omega^c(t) = \mathbb{R}^2 \backslash \Omega(t)$ *and* $\Omega(t) = t\Omega(1) := \{(tx,ty)|(x,y) \in \Omega(1)\}$.
Moreover,

$$\Omega(1) = \{(x,y)|\varphi^*(x,y,1) < 1\}$$

and

$$\varphi^*(x,y,t) := \min_{s\geq 0}\left\{\frac{x^2}{4(t+as)} + \frac{(|y|+s)^2}{4t}\right\}.$$

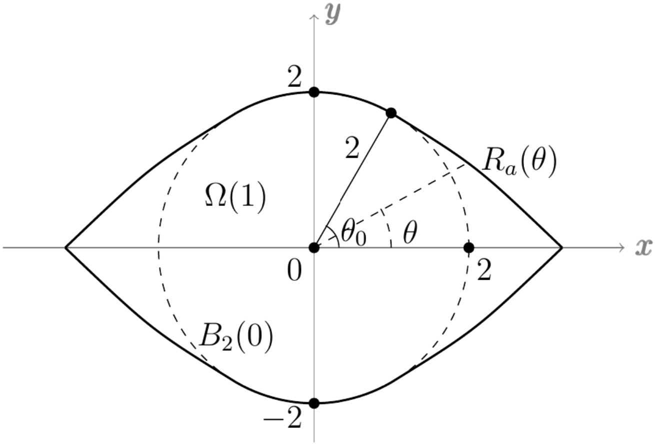
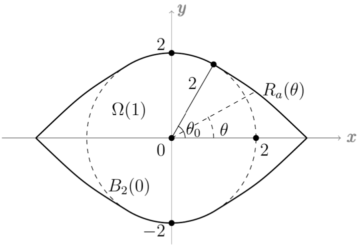


Figure – Asymptotic Spreading Shape $\Omega(1)$



- $R_a(\theta)$: the asymptotic propagation speed along angle θ ;
- $\theta_0 = \arcsin \frac{2a}{1+\sqrt{1+4a^2}}$.
- $\Omega(1)$ is called the **asymptotic expansion shape** :

$$\lim_{t\to\infty} v(xt, yt, t) = \begin{cases} 0, & (x,y) \in \Omega^c(1), \\ 1, & (x,y) \in \Omega(1). \end{cases}$$

Theorem

[a] *There exist* $\Sigma_1, \Sigma_2 \subset \mathbb{R}^2$ *such that*
(i) $\Sigma_1 \subset \Omega_{a_1}(1) \subset \Sigma_2$ *;*
(ii) For each small $\nu > 0$, *the following spreading results hold :*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{\Sigma_2^c((1+\nu)t)} (|v_1| + |v_2|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{\Sigma_2((1-\nu)t)\setminus\Sigma_1((1+\nu)t)} (|v_1| + |v_2 - 1|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{\Sigma_1((1-\nu)t)} (|v_1 - k_1| + |v_2 - k_2|) = 0, \end{cases}$$

where $(k_1, k_2) = \left(\frac{1-b_1}{1-b_1 b_2}, \frac{1-b_2}{1-b_1 b_2}\right)$, $\Sigma_1 = (1-b_1)\Omega_{a_1}(1)$, $\Sigma_2 = \Omega_{a_2}(\sqrt{dr})$.

a. X. Geng and H. Huang, preprint, 2023

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Future Works

- EBCs involving the fractional Laplacian of any order. Suppose

$$A(x)\mathbf{n}(p) = \sigma d(x)^a \mathbf{n}(p),\ A(x)\mathbf{s}(p) = \mu d(x)^a \mathbf{s}(p),$$

where a is a constant ; $d(x)$ is the distance of x onto $\partial\Omega$; p is the unique projection of x on $\partial\Omega$, and $\mathbf{s}(p)$ is an arbitrary tangent vector at p on $\partial\Omega$. Then $A(x)$ is degenerate at the boundary $\partial\Omega$.

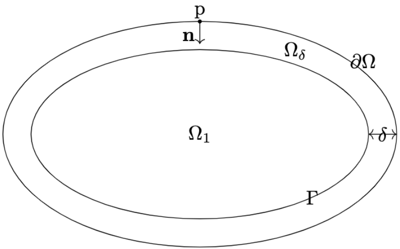

Figure – $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$. Ω is fixed.

Future Works

- Apply the idea of EBCs to the wave equation and the Schrödinger equation, which can provide a physical understanding of the effects of the layer.
- Study the propagation speed for the Fisher-KPP equation on the upper half plane with the boundary condition involving fractional Laplacian of any order.
- Consider the propagation speed for the Fisher-KPP equation on the whole plane with multiple roads, on which a Wentzell-type boundary condition is imposed to enhance the speed.

Publications

1. X. Geng, *Effective boundary conditions arising from the heat equation with three-dimensional interior inclusion*, Comm. Pure Appl. Anal., **22** (2023), 1394-1419.
2. X. Geng, *Effective boundary conditions for heat equation arising from anisotropic and optimally aligned coatings in three dimensions*, arXiv preprint arXiv :2301.13657, (2023).
3. X. Geng, *Effective boundary conditions for the Fisher-KPP equation on a domain with 3-dimensional optimally aligned coatings*, arXiv preprint arXiv :2307.10429, (2023).
4. X. Geng and Y. Wang, *Fractional Laplacian boundary condition as a singular limit of problems degenerating at the boundary*, in preparation.
5. X. Geng and H. Huang, *Asymptotic spreading of competition diffusion systems with an effective boundary condition on a road*, in preparation.

THANK YOU!