Effective Boundary Conditions for 3-dimensional Optimally Aligned Coatings

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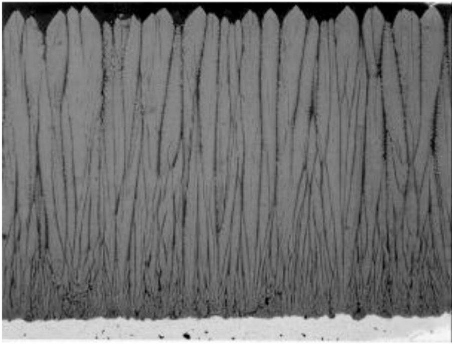
Overview

Motivations : Turbine Engine Blades

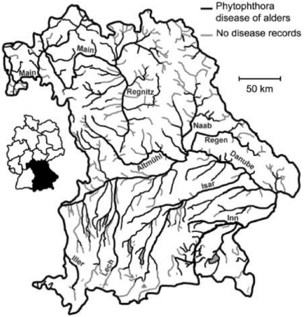
Coatings may be Anisotropic : anisotropy for TBC is caused by the fashions in which the ceramic “YSZ”(yttria-stabilized zirconia) is deposited on the blade :



(Picture taken from J.R. Nicholls K.J. Lawson, A. Johnstone, D.S. Rickerby, 2002)

Motivations : Nature Reserves

Distribution of phytophthora root and collar rot of alders along main rivers and streams in Bavaria. The small map above shows the location of Bavaria within Germany.



(Picture taken from M. Blaschke and T. Jung, 2004)

Motivations : cell

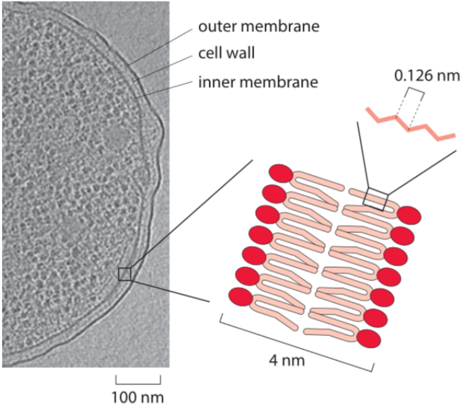


Figure: E. coli cell; membrane thick and diameter ratio 1:500; red-headed molecules are phospholipids

Motivations :

Common features :

- Domain contains a thin component ;
- Diffusion tensors on different components are drastically different.

Issues :

- The multi-scale in size and different diffusion tensors lead to computational difficulty ;
- It is hard to see the effect of the thin component ;

Resolution :

- Think of the thin component as widthless surface and impose "effective boundary conditions"(EBCs).

History

- In 1959, Carlaw and Jaeger, in their classic book *Conduction of Heat in Solids*, first derived EBCs formally ;
- In 1974, rigorous derivation was initiated by Sanchez-Palencia to study the “interior reinforcement problem" for elliptic and parabolic equations with a thin diamond-shaped inclusion layer ;
- In 1980, Brezis, Caffarelli and Friedman studied the elliptic problem in both interior and boundary reinforcement cases ;
- In 1987, Buttazo and Kohn studied the case of thin layer of oscillating thickness ;
- $\cdots$;
- Lots of follow-up work on elastic equations, electromagnetic equations, etc ;

Mathematical model

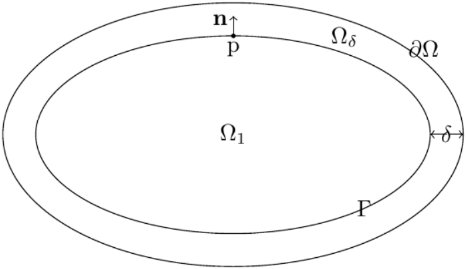


Figure – $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$. The layer Ω_δ is uniformly thick with thickness δ and $\Gamma(= \partial\Omega_1) \in C^2$. Ω_1 is fixed and $\partial\Omega \to \Gamma$as $\delta \to 0$.

Mathematical model

For any fixed and finite $T > 0$, consider

$$\begin{cases} u_t - \nabla \cdot (A(x)\nabla u) = f(x,t), & (x,t) \in Q_T = \Omega \times (0,T), \\ u = 0, & (x,t) \in S_T = \partial\Omega \times (0,T), \\ u = u_0, & (x,t) \in \Omega \times \{0\} \end{cases} \tag{1}$$

where Ω_1 is fixed, $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. $A(x)$ is given by

$$A(x) = \begin{cases} kI_{N\times N}, & x \in \Omega_1 \\ (a_{ij}(x))_{N\times N}, & x \in \Omega_\delta \end{cases} \tag{2}$$

where $k > 0$ and $(a_{ij})_{N\times N}$ is positive-definite. Moreover, u satisfies the transmission conditions in the weak sense

$$u_1 = u_\delta,\ k\frac{\partial u_1}{\partial \mathbf{n}} = A(x)\nabla u_\delta \cdot \mathbf{n} \ \text{ on } \ \partial\Omega_1,$$

where $\mathbf{n}$ is the unit outer norm vector on $\partial\Omega_1$; u_1 and u_δ are the restrictions of u on $\Omega_1 \times (0,T)$ and $\Omega_\delta \times (0,T)$.

Optimally Aligned Coatings

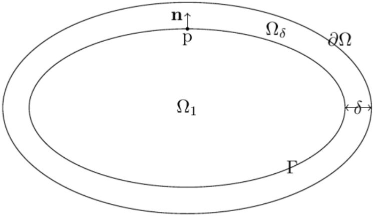


Figure – $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$.

- The notion of optimally aligned coatings [1] is motivated by applications in physics, ecology and cell.
- Optimally aligned coatings : for any $x \in \Omega_\delta$, the normal vector $\mathbf{n}(p)$ is always an eigenvector of $A(x)$.

1. S. Rosencrans and X. Wang, SIAM J. Appl. Math, 2006

Curvilinear coordinates

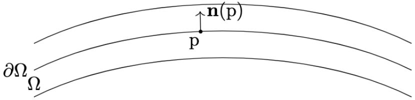


Figure – An illustration of the curvilinear coordinates (p, r).

In the curvilinear coordinates (p, r),

$$x = p + r\mathbf{n}(p), \quad \forall x \in \Omega_\delta,$$

$p-$ projection of x onto $\partial\Omega_1$; $r-$ distance from x to $\partial\Omega_1$.

There are two assumptions of $A(x)$ in Ω_δ.

- Case 1. $A(x)$ has two identical eigenvalues in the tangent directions, i.e.,

$$A(x)\mathbf{n}(p) = \sigma\mathbf{n}(p), \quad A(x)\mathbf{s}(p) = \mu\mathbf{s}(p), \tag{3}$$

  where p is the unique projection of x on $\partial\Omega_1$, and $\mathbf{s}(p)$ is an arbitrary tangent vector at p on $\partial\Omega_1$; $(\sigma, \mu) = (\sigma(\delta), \mu(\delta))$ are two corresponding eigenvalues.

- Case 2. $A(x)$ has two different eigenvalues in the tangent directions, namely, $\partial\Omega_1 = \Gamma_1 \times \Gamma_2$, and $A(x)$ satisfies

$$\begin{aligned} A(x)\mathbf{n}(p) &= \sigma\mathbf{n}(p), \\ A(x)\boldsymbol{\tau}_1(p) &= \mu_1\boldsymbol{\tau}_1(p), \\ A(x)\boldsymbol{\tau}_2(p) &= \mu_2\boldsymbol{\tau}_2(p), \end{aligned} \tag{4}$$

  where $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ are two orthonormal eigenvectors of $A(x)$ in the tangent plane.

Theorem

Assume $A(x)$ *is given by* (2) *and* (3). *Suppose*

$$\lim_{\delta\to 0}\frac{\sigma}{\delta} = \alpha \in [0,\infty] \text{ and } \lim_{\delta\to 0}\sigma\mu = \gamma \in [0,\infty].$$

If u *is the weak solution of* (1), *then as* $\delta \to 0$, $u \to v$ *strongly in* $C([0,T];L^2(\Omega))$, *where* v *is the weak solution of the effective equation :*

$$\begin{cases} v_t - k\Delta v = f(x,t), & (x,t)\in \Omega_1 \times (0,T), \\ v(x,0) = u_0, & x \in \partial\Omega_1, \end{cases} \tag{5}$$

EBCs in 3D

Case 1

Theorem

subject to the EBCs on $\partial\Omega_1 \times (0,T)$ *:*

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu} \to \gamma \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\gamma/\alpha}[v]$	$v = 0$
$\sigma\mu \to \infty$	$\nabla_\Gamma v = 0$, $\int_{\partial\Omega_1} \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_{\partial\Omega_1} (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v)dx = 0$	$v = 0$

Remarks :

- $\nabla_\Gamma v = 0$ means v is a constant in x but may depend on t ;
- $\mathcal{J}_D^H$ is the Dirichlet-to-Neumann mapping ;
- $\mathcal{J}_D^\infty$ is the fractional Laplacian.

In particular, for smooth g defined on $\partial\Omega_1$, and $H \in (0,\infty)$,

$$\mathcal{J}_D^H[g] := \Psi_R(s,0), \tag{6}$$

where Ψ is the solution of

$$\begin{cases} \Psi_{RR} + \Delta_\Gamma \Psi = 0, & \partial\Omega_1 \times (0,H), \\ \Psi(s,0) = g(s), & \Psi(s,H) = 0. \end{cases} \tag{7}$$

Moreover, $\mathcal{J}_D^H$ is linear and symmetric, and its explicit formula can be given in eigenfunctions of the Laplace-Beltrami operator $-\Delta_\Gamma$, from which

$$\mathcal{J}_D^\infty = \lim_{H\to\infty} \mathcal{J}_D^H = -(-\Delta_\Gamma)^{1/2}. \tag{8}$$

Weak solution

Definition

We say that u is a weak solution of (1), if $u(x,t) \in V_{2,0}^{1,0}(Q_T)$ and it holds that

$$\begin{aligned} &\mathcal{A}[u,\xi] \\ =& -\int_\Omega u_0(x)\xi(x,0)dx + \int_{Q_T} (A(x)\nabla u)\cdot\nabla\xi - u\xi_t - f\xi dtdx \\ =&0 \end{aligned} \tag{9}$$

for any $\xi \in W_{2,0}^{1,1}(Q_T)$ satisfying $\xi = 0$ at $t = T$.

Lemma (First order estimates)

$$
\begin{aligned}
&(i) \max_{t\in[0,T]} \int_\Omega u^2(x,t)dx + \int_{Q_T} \nabla u \cdot A(x)\nabla u dxdt \\
&\leq C(T)\Big(\int_\Omega u_0^2 dx + \int_{Q_T} f^2 dxdt\Big); \\
&(ii) \max_{t\in[0,T]} t\int_\Omega \nabla u \cdot A(x)\nabla u(x,t)dx + \int_{Q_T} tu_t^2 dxdt \\
&\leq C(T)\Big(\int_\Omega u_0^2 dx + \int_{Q_T} f^2 dxdt\Big).
\end{aligned} \tag{10}
$$

A priori estimates

Lemma (Second order estimates)

Suppose $\Gamma \in C^3$ *and* $f \in L^2(Q_T)$ *with* $u_0 \in L^2(\Omega)$*. Then, for any fixed* $t_0 > 0$*, the weak solution* u *of (1) satisfies the following inequalities :*

$$\int_{t_0}^{T} \int_{\Omega_\delta} \mu(\Delta_\Gamma u)^2 + \sigma(\nabla_\Gamma u_r)^2 \leq O(1) + O(\frac{\sigma}{\mu}) + O(\frac{1}{\mu}). \tag{11}$$

Lemma (Higher order estimates)

Let $m \geq 2$ *be an integer and* $\alpha \in (0,1)$*. Suppose that* $\partial\Omega_1 \in C^{m+\alpha}$ *and* $f \in C^{m-2+\alpha,(m-2+\alpha)/2}(\overline{\Omega}_h \times [0,T])(h = 1, \delta, 2)$*, and* $a_{ij} \in C^{m-1+\alpha,(m-1+\alpha)/2}(\overline{\Omega}_h \times [0,T])$*, then for any* $t_0 > 0$*, the weak solution* u *of (1) satisfies*

$$u \in C^{m+\alpha,(m+\alpha)/2)}(\overline{\mathcal{N}}_h \times [t_0, T])$$

where $\mathcal{N}$ *is a narrow neighborhood of* $\partial\Omega_1$ *and* $\mathcal{N}_h = \mathcal{N} \cap \Omega_h$*.*

Proof of the theorem

Main Steps :

- Step 1. Existence and uniqueness of weak solution of (1) ;
- Step 2. Energy estimates of the weak solution of (1) and prove $u \to v$ strongly in $C([0,T]; L^2(\Omega))$ as $\delta \to 0$;
- Step 3. Show that v is the weak solution of (5) with related EBCs ;
  - Harmonic extension [2] : construct a test function such that $\overline{\xi}(s,r,t) = \psi(s,r,t)$ in $\overline{\Omega}_\delta$,

$$\begin{cases} \sigma\psi_{rr} + \mu\Delta_\Gamma\psi = 0, & \Gamma \times (0,\delta), \\ \psi(s,0,t) = \xi(s,0,t) & \psi(s,\delta,t) = 0 \end{cases} \tag{12}$$

  - By rescaling, $\Psi(s,R) = \psi(s, \sqrt{\mu/\sigma}r, t)$
- Step 4. Existence and uniqueness of the weak solution of the effective equation (5) with related EBCs ;

2. X. Chen, C. Pond, and Wang, ARMA, 2012

Similarly, if the layer is of interior inclusion, denote

$$A(x) = \begin{cases} k_1, & x \in \Omega_1 \\ (a_{ij})_{N\times N}, & x \in \Omega_\delta \\ k_2, & x \in \Omega_2 \end{cases}$$

where k_1 and k_2 are two positive constants independent of $\delta > 0$.

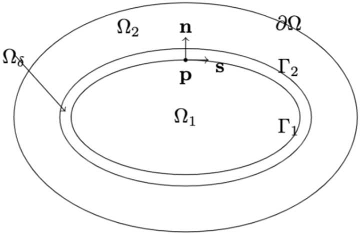


Figure – $\Omega = \Omega_1 \cup \overline{\Omega}_\delta \cup \Omega_2$. Ω and Ω_1 are fixed with $\Gamma_1 \in C^2$.

Denote

$$A_0(x) = \begin{cases} k_1, & x \in \Omega_1, \\ k_2, & x \in \Omega \backslash \overline{\Omega}_1. \end{cases}$$

Suppose $\Gamma_1 \in C^2$, and

$$\lim_{\delta \to 0} \frac{\sigma}{\delta} = b \in [0, \infty], \lim_{\delta \to 0} \sigma\mu = \gamma \in [0, \infty], \lim_{\delta \to 0} \mu\delta = \beta \in [0, \infty].$$

As $\delta \to 0$, $u \to v$ strongly in $C([0,T]; L^2(\Omega))$, where v is the weak solution of

$$\begin{cases} v_t - \nabla \cdot (A_0(x)\nabla v) = f(x,t), & (x,t) \in Q_T, \\ v = 0, & (x,t) \in S_T, \\ v = u_0, & x \in \Omega, t = 0, \end{cases}$$

subject to the effective boundary conditions on $\Gamma_1 \times (0,T)$:

EBCs for interior inclusion in 3D

Case 1. $\frac{\sigma}{\delta} \to 0$ as $\delta \to 0$.

As $\delta \to 0$	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\beta = 0$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$	−−−−−−−	−−−−−−−
$\beta \in (0,\infty)$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$	−−−−−−−	−−−−−−−
$\beta = \infty$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = \gamma \mathcal{J}_1^\infty[v_1]$, $k_2 \frac{\partial v_2}{\partial \mathbf{n}} = -\gamma \mathcal{J}_1^\infty[v_2]$	$\nabla_{\Gamma_1} v_1 = \nabla_{\Gamma_1} v_2 = 0$, $\int_{\Gamma_1} \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $\int_{\Gamma_1} \left(k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}}\right) = 0$

Case 2. $\frac{\sigma}{\delta} \to b \in (0,\infty)$ as $\delta \to 0$.

As $\delta \to 0$	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\beta = 0$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = b(v_2 - v_1)$	−−−−−−−	−−−−−−−
$\beta \in (0,\infty)$	−−−−−−−	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = \gamma \mathcal{J}_1^{\beta/\gamma}[v_1] - \gamma \mathcal{J}_2^{\beta/\gamma}[v_2]$, $k_2 \frac{\partial v_2}{\partial \mathbf{n}} = \gamma \mathcal{J}_2^{\beta/\gamma}[v_1] - \gamma \mathcal{J}_1^{\beta/\gamma}[v_2]$	−−−−−−−
$\beta = \infty$	−−−−−−−	−−−−−−−	$\nabla_{\Gamma_1} v_1 = \nabla_{\Gamma_1} v_2 = 0$, $\int_{\Gamma_1} \left(k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}}\right) = 0$, $\int_{\Gamma_1} \left(k_1 \frac{\partial v_1}{\partial \mathbf{n}} - b(v_2 - v_1)\right) = 0$

EBCs for interior inclusion in 3D

Case 3. $\frac{\sigma}{\delta} \to \infty$ and $\sigma\delta^3 \to 0$ as $\delta \to 0$.

As $\delta \to 0$	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\beta = 0$	$v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$	$v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$	$v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$
$\beta \in (0,\infty)$	−−−−−−−	−−−−−−−	$v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = \beta \Delta_{\Gamma_1} v$
$\beta = \infty$	−−−−−−−	−−−−−−−	$v_1 = v_2$, $\nabla_{\Gamma_1} v = 0$, $\int_{\Gamma_1} \left(k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}}\right) = 0$

Remarks :

- $\sigma\delta^3 \to 0$ can be removed if $\frac{\mu}{\sigma}$ does not vanish as $\delta \to 0$;
- $\nabla_{\Gamma_1} v = 0$ means v is a constant in x but may depend on t ;
- Δ_{Γ_1} is the Laplacian-Beltrami operator ;
- $\mathcal{J}_1^H$, $\mathcal{J}_2^H$ and $\mathcal{J}^H$ are Dirichlet-to-Neumann mappings that can be defined as $\mathcal{J}_D^H$;
- $\mathcal{J}_1^\infty = \mathcal{J}^\infty = -(-\Delta_{\Gamma_1})^{1/2}$ and $\mathcal{J}_2^\infty = 0$.

Theorem

Suppose that Γ *is a topological torus and* $A(x)$ *is given in* (2) *and satisfies* (4). *Let* $u_0 \in L^2(\Omega)$ *and* $f \in L^2(Q_T)$ *with functions being independent of* δ. *Assume further that without loss of generality,* $\mu_1 > \mu_2$. *Moreover,* σ, μ_1, *and* μ_2 *satisfy*

$$\lim_{\delta\to 0} \frac{\mu_2}{\mu_1} = c \in [0,1], \quad \lim_{\delta\to 0} \frac{\sigma}{\delta} = \alpha \in [0,1],$$

$$\lim_{\delta\to 0} \sigma\mu_i = \gamma_i \in [0,\infty], \quad \lim_{\delta\to 0} \mu_i\delta = \beta_i \in [0,\infty], \quad i = 1,2.$$

Theorem

(i) If $c \in (0,1]$*, then as* $\delta \to 0$*,* $u \to v$ *weakly in* $W_2^{1,0}(\Omega_1 \times (0,T))$*, strongly in* $C([0,T]; L^2(\Omega_1))$*, where* v *is the weak solution of (5) subject to the effective boundary conditions :*

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu_1 \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^{\gamma_1/\alpha}[v]$	$v = 0$
$\sigma\mu_1 \to \infty$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_\Gamma (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$	$v = 0$

Figure – *EBCs on* $\partial\Omega_1$ *for* $c = 0$

Theorem

(ii) If $c = 0$ *and* $\lim_{\delta\to 0} \delta^2 \mu_1/\mu_2 = 0$, *then* $u \to v$ *weakly in* $W_2^{1,0}(\Omega_1 \times (0,T))$, *strongly in* $C([0,T]; L^2(\Omega_1))$, *where* v *is the weak solution of* (5) *subject to the effective boundary conditions :*

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu_1 \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^{\gamma_1/\alpha}[v]$	$v = 0$
$\sigma\mu_1 \to \infty$, $\sigma\mu_2 \to 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \frac{\partial v}{\partial \mathbf{n}} = 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(\frac{\partial v}{\partial \mathbf{n}} + \alpha v\right) = 0$	$v = 0$
$\sigma\mu_1 \to \infty$, $\sqrt{\sigma\mu_2} \to \gamma_2 \in (0,\infty)$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^{\infty}[v]\right) = 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^{\gamma_2/\alpha}[v]\right) = 0$	$v = 0$
$\sigma\mu_1 \to \infty$, $\sigma\mu_2 \to \infty$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$v = 0$

Figure – *EBCs on* $\partial\Omega_1 \times (0,T)$ *for* $c = 0$

Exotic EBCs

- $\mathcal{K}_D^H[g](s) := \Phi_R(s,0)$, where Φ is the unique bounded solution of

$$\begin{cases} \Phi_{RR} + \Phi_{s_1 s_1} + c\Phi_{s_2 s_2} = 0, & \Gamma \times (0,H), \\ \Phi(s,0) = g(s), & \Phi(s,H) = 0. \end{cases}$$

- $\Lambda_D^H[g](s) := \Phi_R^0(s,0)$, where Φ^0 is the unique bounded solution of

$$\begin{cases} \Phi_{RR}^0 + \Phi_{s_1 s_1}^0 = 0, & \Gamma \times (0,H), \\ \Phi^0(s,0) = g(s), & \Phi^0(s,H) = 0. \end{cases}$$

- $\mathcal{D}_D^H[g](s_2) := \Phi_R(s_2,0)$, where $\Phi(s_2,R)$ is the bounded solution of

$$\begin{cases} \Phi_{RR} + \Phi_{s_2 s_2} = 0, & \Gamma_2 \times (0,H), \\ \Phi(s_2,0) = g(s_2), & \Phi(s_2,H) = 0. \end{cases}$$

Ongoing Work

- It is natural to find the EBC involving the fractional Laplacian of any order.
- Suppose

$$A(x)\mathbf{n}(p) = \sigma d(x)^a \mathbf{n}(p),$$
$$A(x)\mathbf{s}(p) = \mu d(x)^a \mathbf{s}(p),$$

where a is a constant ; $d(x)$ is the distance of x onto $\partial\Omega$; p is the unique projection of x on $\partial\Omega$, and $\mathbf{s}(p)$ is an arbitrary tangent vector at p on $\partial\Omega$.

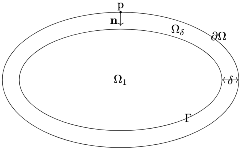


Figure – $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$. Ω and Ω are fixed.

THANK YOU!