# Effective Boundary Conditions for the Heat Equation with Three-dimensional Anisotropic and Optimally Aligned Coatings

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# Overview

# Motivations : cell

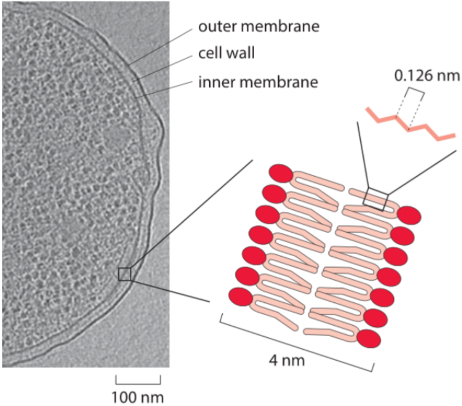
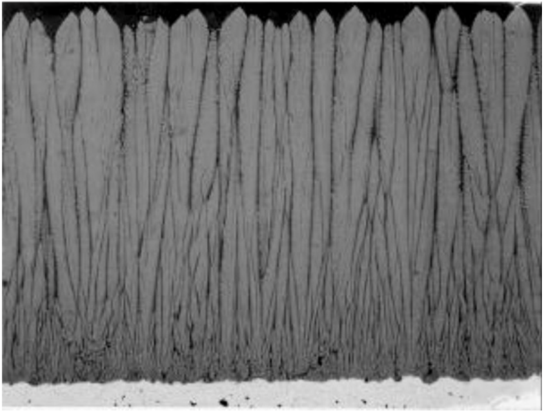



Figure: E. coli cell; membrane thick and diameter ratio 1:500; red-headed molecules are phospholipids

Coatings may be Anisotropic : anisotropy for TBC is caused by the fashions in which the ceramic “YSZ”(yttria-stabilized zirconia) is deposited on the blade :

# TBC

If YSZ is sprayed on by electron beam physical vapor deposition (EB-PVD) method, then parallel crystal columns that are perpendicular to the boundary form ; and between these columns a small volume fraction of elongated pores also form. (Picture taken from J.R. Nicholls K.J. Lawson, A. Johnstone, D.S. Rickerby)



# Motivations :

Common features :

- Domain contains a thin component ;
- Diffusion tensors on different components are drastically different.

Issues :

- The multi-scale in size and different diffusion tensors lead to computational difficulty ;
- It is hard to see the effect of the thin component ;

Resolution :

- Think of the thin component as widthless surface and impose "effective boundary conditions"(EBCs).

# History

- As early as 1959, Carlaw and Jaeger, in their classic book *Conduction of Heat in Solids*, first derived EBCs formally ;
- Rigorous derivation was initiated by Sanchez-Palencia in 1974, to study Laplace equation and the heat equation with thin diamond-shaped inclusions ;
- In 1980, Brezis, Caffarelli and Friedman studied the case of Poisson equation ;
- In 1987, Buttazo and Kohn studied the case of thin layer of oscillating thickness ;
- Lots of follow-up work on elastic equations, electromagnetic equations, etc ;

# Mathematical model

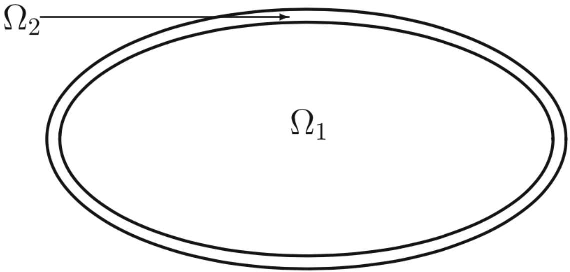



**Fig. 1.** The domain $\Omega = \overline{\Omega}_1 \cup \Omega_2 \subset \mathbb{R}^N$ consists of an isotropic body $\Omega_1$ surrounded by a layer $\Omega_2$ of uniform thickness $\delta$

# Mathematical model

For any fixed $T > 0$,

$$\begin{cases} u_t - \nabla \cdot (A(x)\nabla u) = f(x,t), & (x,t) \in Q_T = \Omega \times (0,T), \\ u = 0, & (x,t) \in S_T = \partial\Omega \times (0,T), \\ u = u_0, & (x,t) \in \Omega \times \{0\} \end{cases} \quad (1)$$

where $\Omega_1$ is fixed, $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. $A(x)$ is given by

$$A(x) = \begin{cases} kI_{N\times N}, & x \in \Omega_1 \\ (a_{ij}(x))_{N\times N}, & x \in \Omega_2 \end{cases} \quad (2)$$

where $k$ is a positive constant and $(a_{ij})_{N\times N}$ is positive-definite. Moreover, $u$ satisfies

$$u_1 = u_2,\ k\frac{\partial u_1}{\partial \mathbf{n}} = A(x)\nabla u_2 \cdot \mathbf{n} \ \text{ on } \ \partial\Omega_1,$$

where $\mathbf{n} = (n_1, ..., n_N)$ is the unit outer norm vector on $\partial\Omega_1$.

# Results by Li, Rosencrans, Zhang and Wang [1]

- Suppose $a_{ij}(x) = \sigma(\overline{a}_{ij}(x))$ with $\overline{a}_{ij}(x) \in C^1(\overline{\Omega}_2)$ and $\sigma(\delta)$ is a positive parameter.
- If $\sigma$ is bounded and $\lim_{\delta\to 0}\frac{\sigma}{\delta} = \alpha$, then $u \to v$ in $L^2(\Omega_1 \times [0,T])$, where $v$ is the weak solution of

$$\begin{cases} v_t - k\Delta v = f(x,t), & (x,t) \in Q_T, \\ k\frac{\partial v}{\partial \mathbf{n}} + \alpha(\sum_{i,j}\overline{a}_{ij}(x)n_i n_j)v = 0, & (x,t) \in S_T, \\ v = u_0, & x \in \Omega, t = 0 \end{cases} \tag{3}$$

- A nature question : what is the effective boundary condition if $\sigma \to \infty$ as $\delta \to 0$ ?

1. J. Li, S. Rosencrans, K. Zhang, and X. Wang, 2009, Proc. AMS

# Interior Inclusion in 3D

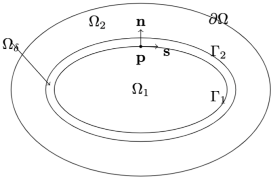


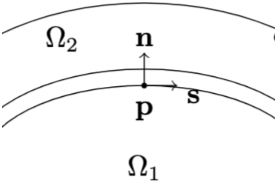

Figure.1: $\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_\delta \cup \Omega_2$.

Let

$$A(x) = \begin{cases} k_1, & x \in \Omega_1 \\ (a_{ij})_{3\times 3}, & x \in \Omega_\delta \\ k_2, & x \in \Omega_2 \end{cases}$$

where $k_1$ and $k_2$ are two positive constants independent of $\delta > 0$; $\sigma$ is a positive function of $\delta$; $\Omega$ and $\Omega_1$ are fixed with $\Gamma_1 \in C^2$.

# Optimally aligned condition




Optimally aligned coating [2] : for any $x \in \Omega_\delta$, $\mathbf{n}(p)$ is always an eigenvector. In the curvilinear coordinates $(s, r)$, $x = p + r\mathbf{n}(p) \in \Omega_\delta$, suppose

$$A(x)\mathbf{n}(p) = \sigma \mathbf{n}(p), \quad A(x)\mathbf{s}(p) = \mu \mathbf{s}(p), \tag{4}$$

where $\mathbf{p}-$ the projection of $x$ on $\Gamma_1 = \partial\Omega_1$ ; $r-$ distance from $x$ to $\Gamma_1$ ; $\mathbf{s}(p)$ is an arbitrary tangent vector at $p$ on $\partial\Omega_1$.

2. S. Rosencrans, and X. Wang, SIAM J. Appl. Math, 2006

# Sobolev spaces

$$W_2^{1,0}(Q_T) = \{u \in L^2(Q_T) : \nabla u \in L^2(Q_T)\};$$
$$W_{2,0}^{1,0}(Q_T) = \{u \in W_2^{1,0}(Q_T) : \text{ with trace 0 on} S_T\};$$
$$W_2^{1,1}(Q_T) = \{u \in L^2(Q_T) : u_t, \nabla u \in L^2(Q_T)\};$$
$$W_{2,0}^{1,1}(Q_T) = \{u \in W_2^{1,1}(Q_T) : \text{ with trace 0 on} S_T\};$$
$$V_2^{1,0}(Q_T) = \{u \in W_2^{1,0}(Q_T) : u \in C([0,T], L^2(\Omega))\};$$
$$V_{2,0}^{1,0}(Q_T) = \{u \in V_2^{1,0}(Q_T) : \text{ with trace 0 on} S_T\};$$

# Weak solutions

## Definition

$u$ is a weak solution of (1), if $u(x,t) \in V_{2,0}^{1,0}(Q_T)$ and it holds that

$$\begin{aligned} &\mathcal{A}[u,\xi] \\ =& -\int_\Omega u_0(x)\xi(x,0)dx + \int_{Q_T} (A(x)\nabla u)\cdot\nabla\xi - u\xi_t - f\xi dtdx \\ =&0 \end{aligned} \tag{5}$$

for any $\xi \in W_{2,0}^{1,1}(Q_T)$ satisfying $\xi = 0$ at $t = T$,

# *A priori* estimates

**Lemma**

$$
\begin{aligned}
&(i)\ \max_{t\in[0,T]}\int_\Omega u^2(x,t)dx+\int_{Q_T}\nabla u\cdot A(x)\nabla u dxdt\\
&\le C(T)(\int_\Omega u_0^2dx+\int_{Q_T}f^2dxdt);\\
&(ii)\ \max_{t\in[0,T]} t\int_\Omega \nabla u\cdot A(x)\nabla u(x,t)dx+\int_{Q_T}tu_t^2dxdt\\
&\le C(T)(\int_\Omega u_0^2dx+\int_{Q_T}f^2dxdt).
\end{aligned}
\tag{6}
$$

## Lemma (2)

*Suppose* $\Gamma_1 \in C^3$ *and* $f \in L^2(Q_T)$ *with* $u_0 \in L^2(\Omega)$. *Then, for any fixed* $t_0 > 0$, *the weak solution* $u$ *of (1) satisfies the following inequalities :*

$$\int_{t_0}^{T} \int_{\Omega_\delta} \mu(\Delta_\Gamma u)^2 + \sigma(\nabla_\Gamma u_r)^2 \leq O(1) + O(\frac{\sigma}{\mu}) + O(\frac{1}{\mu}) \tag{7}$$

*and*

$$\int_{t_0}^{T} \int_{\Omega_\delta} \sigma u_{rr}^2 \leq O(1) + O(\frac{1}{\sigma}) + O(\frac{\mu}{\sigma}) \tag{8}$$

# General results[3]

**Theorem (X.Chen, C.Pond and X.Wang)**

*Let $m \geq 2$ be an integer and $\alpha \in (0,1)$. Suppose that $\partial\Omega_1 \in C^{m+\alpha}$ and $f \in C^{m-2+\alpha,(m-2+\alpha)/2}(\overline{\Omega}_h \times [0,T])(h = 1, \delta, 2)$, and $a_{ij} \in C^{m-1+\alpha,(m-1+\alpha)/2}(\overline{\Omega}_h \times [0,T])$, then for any $t_0 > 0$, the weak solution $u$ of (1) satisfies*

$$u \in C^{m+\alpha,(m+\alpha)/2)}(\overline{\mathcal{N}}_h \times [t_0, T])$$

*where $\mathcal{N}$ is a narrow neighborhood of $\partial\Omega_1$ and $\mathcal{N}_h = \mathcal{N} \cap \Omega_h$.*

3. X.Chen, C.Pond and X.Wang, Arch.Ration.Mech.Anal.(2012)

# Main results

## Theorem (Geng)

[a] *Denote* $A_0(x) = k_1, x \in \Omega_1$ *and* $A_0 = k_2, x \in \Omega \backslash \overline{\Omega}_1$. *Suppose* $\Gamma_1 \in C^3$

$$\lim_{\delta \to 0} \frac{\sigma}{\delta} = b \in [0, \infty], \quad \lim_{\delta \to 0} \sigma\delta = a \in [0, \infty],$$
$$\lim_{\delta \to 0} \sigma\mu = \gamma \in [0, \infty], \quad \lim_{\delta \to 0} \mu\delta = \beta \in [0, \infty]. \tag{9}$$

*As* $\delta \to 0$, $u \to v$ *strongly in* $C([0, T]; L^2(\Omega))$, *where* $v$ *is the weak solution of*

$$\begin{cases} v_t - \nabla \cdot (A_0(x)\nabla v) = f(x, t), & (x, t) \in Q_T, \\ v = 0, & (x, t) \in S_T, \\ v = u_0, & x \in \Omega, t = 0, \end{cases} \tag{10}$$

*subject to the effective boundary conditions on* $\Gamma_1 \times (0, T)$ :

a. Xingri Geng, submitted

# Main results

## Theorem (Geng)

Case 1. $\frac{\sigma}{\delta} \to 0$ as $\delta \to 0$.

| As $\delta \to 0$ | $\gamma = 0$ | $\gamma \in (0, \infty)$ | $\gamma = \infty$ |
|---|---|---|---|
| $\beta = 0$ | $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ | – – – – – – | – – – – – – |
| $\beta \in (0, \infty)$ | $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ | – – – – – – | – – – – – – |
| $\beta = \infty$ | $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ | $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = \gamma \mathcal{J}_1^\infty[v_1]$, $k_2 \frac{\partial v_2}{\partial \mathbf{n}} = -\gamma \mathcal{J}_1^\infty[v_2]$ | $\nabla_{\Gamma_1} v_1 = \nabla_{\Gamma_1} v_2 = 0$, $\int_{\Gamma_1} \frac{\partial v_1}{\partial \mathbf{n}} = 0$, $\int_{\Gamma_1} \left(k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}}\right) = 0$ |

Case 2. $\frac{\sigma}{\delta} \to b \in (0, \infty)$ as $\delta \to 0$.

| As $\delta \to 0$ | $\gamma = 0$ | $\gamma \in (0, \infty)$ | $\gamma = \infty$ |
|---|---|---|---|
| $\beta = 0$ | $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = b(v_2 - v_1)$ | – – – – – – | – – – – – – |
| $\beta \in (0, \infty)$ | – – – – – – | $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = \gamma \mathcal{J}_1^{\beta/\gamma}[v_1] - \gamma \mathcal{J}_2^{\beta/\gamma}[v_2]$, $k_2 \frac{\partial v_2}{\partial \mathbf{n}} = \gamma \mathcal{J}_2^{\beta/\gamma}[v_1] - \gamma \mathcal{J}_1^{\beta/\gamma}[v_2]$ | – – – – – – |
| $\beta = \infty$ | – – – – – – | – – – – – – | $\nabla_{\Gamma_1} v_1 = \nabla_{\Gamma_1} v_2 = 0$, $\int_{\Gamma_1} \left(k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}}\right) = 0$, $\int_{\Gamma_1} \left(k_1 \frac{\partial v_1}{\partial \mathbf{n}} - b(v_2 - v_1)\right) = 0$ |

# Main results

**Theorem (Geng)**

Case 3. $\frac{\sigma}{\delta} \to \infty$ and $\sigma\delta^3 \to 0$ as $\delta \to 0$.

| As $\delta \to 0$ | $\gamma = 0$ | $\gamma \in (0, \infty)$ | $\gamma = \infty$ |
|---|---|---|---|
| $\beta = 0$ | $v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ | $v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ | $v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ |
| $\beta \in (0, \infty)$ | $- - - - - --$ | $- - - - - --$ | $v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = \beta \Delta_{\Gamma_1} v$ |
| $\beta = \infty$ | $- - - - - --$ | $- - - - - --$ | $v_1 = v_2$, $\nabla_{\Gamma_1} v = 0$, $\int_{\Gamma_1} \left( k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} \right) = 0$ |

# Exotic EBCs

Remarks :

- $\sigma\delta^3 \to 0$ can be removed if $\frac{\mu}{\sigma}$ does not vanish as $\delta \to 0$ ;
- $\nabla_{\Gamma_1} v = 0$ means $v$ is a constant in $x$ but may depend on $t$ ;
- $\Delta_{\Gamma_1}$ is the Laplacian-Beltrami ;
- $\mathcal{J}^H$ is called Dirichlet-to-Neumann map ;
- Lots of new and exotic EBCs emerge, including the fractional Laplacian-Beltrami and the Dirichlet-to-Neumann mapping.

Let $g(s)$ be a function on $\partial\Omega_1$ and $\Psi$ is defined by

$$\begin{cases} \Psi_{RR} + \Delta_\Gamma \Psi = 0, & \partial\Omega_1 \times (0, H), \\ \Psi(s, 0) = g(s), & \Psi(s, H) = g(s). \end{cases} \tag{11}$$

Moreover,

$$\mathcal{J}^H[g] := \Psi_R(s, 0), \tag{12}$$

where $\mathcal{J}^H$ is symmetric for $H \in (0, \infty)$ and

$$\mathcal{J}^\infty = \lim_{H \to \infty} \mathcal{J}^H = -(-\Delta_{\Gamma_1})^{1/2}. \tag{13}$$

$\mathcal{J}^H$ can be given in eigenfunctions of the Laplace-Beltrami operator.

# Specical case : $\sigma = \mu$

## Theorem (Isotropic case)

*(i) If* $\lim_{\delta\to 0}\frac{\sigma}{\delta} = b \in [0,\infty]$ *and* $\sigma \to 0$, *then*

$$k_1\frac{\partial v_1}{\partial \boldsymbol{n}} = b(v_1 - v_2), \quad k_1\frac{\partial v_1}{\partial \boldsymbol{n}} = k_2\frac{\partial v_2}{\partial \boldsymbol{n}}.$$

*(ii)If* $\lim_{\delta\to 0}\sigma\delta = a \in [0,\infty)$ *and* $\sigma \geq O(1) > 0$, *then*

$$v_1 = v_2, \quad k_1\frac{\partial v_1}{\partial \boldsymbol{n}} - k_2\frac{\partial v_2}{\partial \boldsymbol{n}} = a\Delta_{\Gamma_1} v.$$

*(iii) If* $\lim_{\delta\to 0}\sigma\delta = \infty$, *then*

$$v_1 = v_2, \quad \nabla_{\Gamma_1} v = 0, \quad \int_{\Gamma_1}\left(k_1\frac{\partial v_1}{\partial \boldsymbol{n}} - k_2\frac{\partial v_2}{\partial \boldsymbol{n}}\right)ds = 0,$$

*where* $v_1$ *and* $v_2$ *are the restrictions of* $v$ *on* $\Omega_1 \times (0,T)$ *and* $(\Omega\backslash\Omega_1)\times(0,T)$, *respectively.*

# Proof of the theorem

Main Steps :

- Step 1 : existence and uniqueness of weak solution of (1) ;
- Step 2 : energy estimates of the weak solution of (1) and prove $u \to v$ strongly in $C([0,T]; L^2(\Omega))$ as $\delta \to 0$ ;
- Step 3 : show that $v$ is the exact weak solution of (10) with related EBCs ;
  - Construct a test function such that $\bar{\xi}(s,r,t) = \psi(s,r,t)$ in $\Omega_\delta$,

$$\begin{cases} \sigma\psi_{rr} + \mu\Delta_\Gamma\psi = 0, & \Gamma_1 \times (0,\delta), \\ \psi(s,0,t) = g_1(s) & \psi(s,\delta,t) = g_2(s) \end{cases} \tag{14}$$

  - By rescaling, $\Psi(s,R) = \psi(s, \sqrt{\mu/\sigma}r, t)$
- Step 4 : existence and uniqueness of the weak solution of the effective equation (10) with related EBCs ;

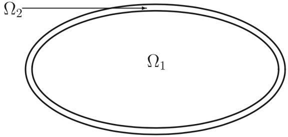



**Fig. 1.** The domain $\Omega = \overline{\Omega}_1 \cup \Omega_2 \subset \mathbb{R}^N$ consists of an isotropic body $\Omega_1$ surrounded by a layer $\Omega_2$ of uniform thickness $\delta$

### Theorem (Boundary case)

[a] *Suppose $A(x)$ is given by (4). If $u$ is the weak solution of (1), then as $\delta \to 0$, $u \to v$ strongly in $C([0,T]; L^2(\Omega))$, where $v$ is the weak solution of the effective equation :*

$$\begin{cases} v_t - k\Delta v = f(x,t), & (x,t) \in \Omega_1 \times (0,T), \\ v(x,0) = u_0, & x \in \partial\Omega_1, \end{cases} \tag{15}$$

a. Xingri Geng, Submitted

## Theorem (Boundary case)

*subject to the following effective boundary conditions :*

| As $\delta \to 0$ | $\frac{\sigma}{\delta} \to 0$ | $\frac{\sigma}{\delta} \to \alpha \in (0, \infty)$ | $\frac{\sigma}{\delta} \to \infty$ |
|---|---|---|---|
| $\sigma\mu \to 0$ | $\frac{\partial v}{\partial \mathbf{n}} = 0$ | $k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$ | $v = 0$ |
| $\sqrt{\sigma\mu} \to \gamma \in (0, \infty)$ | $k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\infty}[v]$ | $k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\gamma/\alpha}[v]$ | $v = 0$ |
| $\sigma\mu \to \infty$ | $\nabla_\Gamma v = 0$, $\int_{\partial\Omega_1} \frac{\partial v}{\partial \mathbf{n}} = 0$ | $\nabla_\Gamma v = 0$, $\int_{\partial\Omega_1} (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v)dx = 0$ | $v = 0$ |

# Boundary layers

## Two different tangent thermal conductivity

Assume $\partial\Omega_1$ is a topological torus, namely, $\partial\Omega_1 = \Gamma_1 \times \Gamma_2$, and $A(x)$ satisfies

$$
\begin{aligned}
A(x)\mathbf{n}(p) &= \sigma \mathbf{n}(p), \\
A(x)\mathbf{t}_1(p) &= \mu_1 \mathbf{t}_1(p), \\
A(x)\mathbf{t}_2(p) &= \mu_2 \mathbf{t}_2(p),
\end{aligned}
\tag{16}
$$

where $\mathbf{t}_1, \mathbf{t}_2$ are two orthonormal eigenvectors of $A(x)$ in the tangent plane. WOLG, suppose $\mu_1 > \mu_2$ and $\frac{\mu_2}{\mu_1} \to c \in [0,1]$.

- If $c \in (0,1]$, EBCs are similar as above ;
- If $c = 0$ with $\frac{\mu_2/\mu_1}{\delta^2} \to 0$, new results arise.

# EBCs

| As $\delta \to 0$ | $\frac{\sigma}{\delta} \to 0$ | $\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$ | $\frac{\sigma}{\delta} \to \infty$ |
|---|---|---|---|
| $\sigma\mu_1 \to 0$ | $\frac{\partial v}{\partial \mathbf{n}} = 0$ | $k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$ | $v = 0$ |
| $\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$ | $k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^\infty[v]$ | $k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^{\gamma_1/\alpha}[v]$ | $v = 0$ |
| $\sigma\mu_1 \to \infty$ | $\nabla_\Gamma v = 0,$ $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$ | $\nabla_\Gamma v = 0,$ $\int_\Gamma (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$ | $v = 0$ |

Figure – EBCs on $\partial\Omega_1$ for $c \neq 0$

| As $\delta \to 0$ | $\frac{\sigma}{\delta} \to 0$ | $\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$ | $\frac{\sigma}{\delta} \to \infty$ |
|---|---|---|---|
| $\sigma\mu_1 \to 0$ | $\frac{\partial v}{\partial \mathbf{n}} = 0$ | $k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$ | $v = 0$ |
| $\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$ | $k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^\infty[v]$ | $k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^{\gamma_1/\alpha}[v]$ | $v = 0$ |
| $\sigma\mu_1 \to \infty$, $\sigma\mu_2 \to 0$ | $\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0,$ $\int_{\Gamma_1} \frac{\partial v}{\partial \mathbf{n}} = 0$ | $\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0,$ $\int_{\Gamma_1} (\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$ | $v = 0$ |
| $\sigma\mu_1 \to \infty$, $\sqrt{\sigma\mu_2} \to \gamma_2 \in (0,\infty)$ | $\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0,$ $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^\infty[v]\right) = 0$ | $\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0,$ $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^{\gamma_2/\alpha}[v]\right) = 0$ | $v = 0$ |
| $\sigma\mu_1 \to \infty$, $\sigma\mu_2 \to \infty$ | $\nabla_\Gamma v = 0,$ $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$ | $\nabla_\Gamma v = 0,$ $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$ | $v = 0$ |

Figure – EBCs on $\partial\Omega_1$ for $c = 0$

## Dirichlet-to-Neumann map

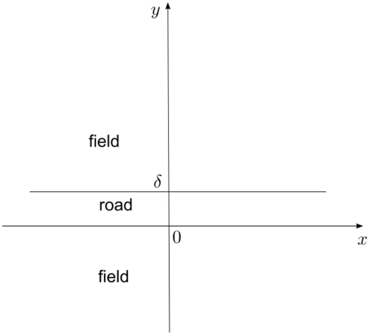
For $H \in (0, \infty)$, $\Lambda_D^H[g] := \Phi_R(s, 0)$ and $\Lambda_D^\infty := \lim_{H\to\infty} \Lambda_D^H$, where $\Phi(s, R)$ is the bounded solution of

$$\begin{cases} \Phi_{RR} + \Phi_{s_1 s_1} = 0, & \partial\Omega_1 \times (0, H), \\ \Phi(s, 0) = g(s), & \Phi(s, H) = 0. \end{cases} \tag{17}$$

Moreover, $\mathcal{D}_D^H[g] := \Psi_R(s_2, 0)$ and $\mathcal{D}_D^\infty := \lim_{H\to\infty} \mathcal{D}_D^H$, where $\Psi(s_2, R)$ is the bounded solution of

$$\begin{cases} \Psi_{RR} + \Psi_{s_2 s_2} = 0, & \Gamma_2 \times (0, H), \\ \Psi(s_2, 0) = g(s_2), & \Psi(s_2, H) = 0. \end{cases} \tag{18}$$

# Ongoing Work




- Use the idea of EBCs to derive new model for other nonlinear equations such as the Fisher-KPP equation and reaction diffusion systems.

THANK YOU !